

REALIZATIONS VIA PREORDERINGS WITH APPLICATION TO THE SCHUR CLASS

MICHAEL A. DRITSCHER

In loving memory of Deborah Godsey

ABSTRACT. We extend Agler's notion of a function algebra defined in terms of test functions to include products, in analogy with the practice in real algebraic geometry, and hence the term preordering in the title. This is done over abstract sets and no additional property, such as analyticity, is assumed. Realization theorems give several equivalent ways of characterizing the unit ball (referred to as the Schur-Agler class) of the function algebras. These typically include, in Agler's terminology, a model (here called an Agler decomposition), a transfer function representation, and an analogue of the von Neumann inequality. The new ingredient is a certain set of matrix valued functions termed "auxiliary test functions" used in constructing transfer functions. In important cases, the realization theorems can be strengthened so as to allow applications to Pick type interpolation problems, among other things. Principle examples have as the domain the polydisk \mathbb{D}^d . The algebras then include $H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$ (where the unit ball is traditionally called the Schur class) and $A(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$, the multivariable analogue of the disk algebra. As an application, it is shown that over the polydisk \mathbb{D}^d , (weakly continuous) representations which are 2^{d-2} contractive are completely contractive (hence having a commuting unitary dilation), offering fresh insight into such examples as Parrott's of contractive representations of $A(\mathbb{D}^3)$ which are not completely contractive.

1. INTRODUCTION

The classical realization theorem gives a variety of characterizations of those functions which are in the Schur class over the unit disk \mathbb{D} in the complex plane \mathbb{C} ; that is, those functions in the closed unit ball of $H^\infty(\mathbb{D})$.

Jim Agler found a method for extending this result to the polydisk \mathbb{D}^d [2], though for dimension d greater than 2, one must use a different norm than the H^∞ norm over an algebra of functions which potentially may be a proper subalgebra of $H^\infty(\mathbb{D}^d)$. The unit ball for such an algebra is now commonly known as the Schur-Agler class; the term Schur class usually being reserved for the unit ball of $H^\infty(X)$ when X is a domain in \mathbb{C}^d . Among other things, the realization theorem states that a complex function φ on the polydisk is in the Schur-Agler class if and only if it has a so-called Agler decomposition, expressing $1 - \varphi\varphi^*$ as an element of a cone generated by products of certain positive kernels and kernels of the form $1 - \psi\psi^*$, where ψ is a coordinate function. Other equivalent conditions for membership in the Schur-Agler class include the existence of a transfer function representation and a von Neumann type inequality for suitably restricted tuples of commuting contractions. The equivalence of all of these conditions makes no a priori assumptions about the function φ , and it is this which enables the use of the realization theorem in such applications as Pick interpolation.

Date: January 20, 2016.

2010 Mathematics Subject Classification. 47A13 (Primary), 47A57, 46J10, 47L55, 47A20, 47A25, 46E22 (Secondary).

Key words and phrases. Realizations, preorderings, Schur class, Schur-Agler class, Pick interpolation, boundary representations, rational dilation.

Part of this work was carried out during several visits to the Indian Institute of Science in Bangalore. The author gives special thanks to Tirtha Bhattacharyya and Gadadhar Misra for their warm hospitality and stimulating conversations.

These results have been vastly generalized, in the spirit of Agler's work (see, for example, [7, 13, 15–18, 26, 33, 34, 37, 45]). In particular, for a planar domain X with smooth boundary components, the necessary machinery has been developed to enable Agler's methods to be applied to the study of $H^\infty(X)$, and as a result, the settling of long-standing problems in operator theory related to these domains [3, 27, 28, 41]. When X is the polydisk \mathbb{D}^d , $d \geq 3$, far less is known, though important initial steps have been taken [30, 35]. A central goal of this paper is to give realization theorems for $H^\infty(\mathbb{D}^d)$, and to apply it to elucidating certain phenomena observed in the study of commuting tuples of contraction operators. Following [28], this is done abstractly, with the results for the polydisk constituting a special example. Indeed, the results cover a wide range of domains and function algebras on these domains, while at the same time demonstrating that assumptions of such properties as analyticity are in general unnecessary.

The Agler decomposition has its analogues in real algebraic geometry. For example, if we have a set in \mathbb{R}^n consisting of those points at which a finite collection of polynomials is non-negative (a so-called basic semi-algebraic set), and these polynomials also include $1 - \psi_i^2$ where each ψ_i is a constant multiples of a coordinate function, then Putinar's theorem [43] (see also [42]) states that a strictly positive polynomial is in the quadratic module generated by these polynomials; that is, it is in the cone generated by finite sums of squares of polynomials times the individual polynomials defining the semi-algebraic set. If however the polynomials $1 - \psi_i^2$ are not necessarily included, the statement of Putinar's theorem is in general false, even if the semi-algebraic set is assumed to be compact [42]. However the situation can be salvaged in the compact setting by replacing the quadratic module by a preordering; that is, by considering the cone generated by finite sums of squares of polynomials multiplied by the various products of the polynomials defining the semi-algebraic set. This is the content of Schmüdgen's theorem [44]. Further refinements are possible. For example, if only two polynomials define the compact semi-algebraic set then one can get by with the quadratic module in Schmüdgen's theorem ([42, Corollary 6.3.7]), which because of Andô's theorem is analogous to what happens in the complex case with Agler's realization theorem.

Back in the complex function setting, work of Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman [30] shows that on the polydisk for dimension greater than 2, one can recover the entire Schur class by using the appropriate variant of a preordering (see also Knese [35]). The caveat is that they find it necessary to assume that the function they are considering is already known to be in the Schur class, and so there is no direct application to Pick interpolation in the Schur class.

Another hurdle to using the results in [30] for interpolation is that the crucial transfer function representation is absent, though they do prove that a form of the von Neumann inequality is available. A particularly interesting aspect of [30] is that the tuples of operators the authors are considering have a unitary dilation, obtained by showing that they induce a completely contractive representation of $H^\infty(\mathbb{D}^d)$ and then applying the standard machinery. There are many papers which consider the problem of determining conditions under which a tuple of commuting contractions has a unitary dilation, including those of Ball, Li, Timotin and Trent [14] and Archer [9], which prove a multivariable form of the commutant lifting theorem.

This paper has several goals. The first is to place the work of Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman in the context of test functions on a set X , in this way allowing for a much broader class of function algebras. For example, there will be analogues, written $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, of the algebra $H^\infty(\mathbb{D})$. The set X can be topologized and closed in an appropriate norm, which allows us to make sense of the analogues $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ of the disk algebra $A(\mathbb{D})$, in this

context; that is, elements $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ which extend continuously to the closure of X . It is noteworthy that there are *a priori* no assumptions made on the set X or on the set of test functions (such as analyticity).

To begin with, a careful examination of the continuity properties of elements of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ and $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ is carried out. Following this, we introduce the auxiliary test functions. In contrast to the original test functions, which are taken to be scalar valued, these are matrix valued functions. Moreover, in the setting of the so-called standard ample preordering, the auxiliary test functions can be taken to be functions in matrix valued $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, and the Schur-Agler class corresponding to these functions is the unit ball of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. As a consequence, we are then able to give a full version of the realization theorem, including the transfer function representation and analogues of the von Neumann inequality. Importantly, none of the realization theorems requires the assumption that the function under consideration is already in $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. Thus in principle, in the ample case such applications as Agler-Pick interpolation are possible.

Even if the preordering is not ample, we show that elements of our generalized Schur-Agler class have a transfer functions representation, though it is not clear that everything with a transfer function representation is in our algebra except in the ample and the classical settings. However, the transfer functions with values in $\mathcal{L}(\mathcal{H})$ do form the unit ball of an algebra having a natural matrix norm structure, and so form an operator algebra. This nicely complements work in [37], where it is shown that a collection of analytic (potentially matrix valued) test functions over a domain in \mathbb{C}^n generate an operator algebra, and that a transfer function representation exists for the functions in this algebra; that is, such algebras are examples of transfer function algebras.

We are able to show that for transfer function algebras, certain types of representations (the so-called Brehmer representations over the analogue of the disk algebra and the weakly continuous Brehmer representations over the analogue of H^∞) are completely contractive, implying the existence of a dilation of such a representation to something akin to a boundary representation (though without the assumption of irreducibility). This includes those representations which are contractive on the auxiliary test functions in the ample setting when we know these functions are in a matrix version of $A(X, \mathcal{K}_\Lambda)$, meaning that such representations are also completely contractive. As a consequence, any representation which is n -contractive for appropriate n (depending only on the number of test functions) will be completely contractive for $A(X, \mathcal{K}_\Lambda)$. Curiously, the condition of being a Brehmer representation does not obviously imply that the representation is contractive on auxiliary test functions, though this is ultimately an outcome of the realization theorems.

As mentioned above, the case when X is the polydisk is of particular interest. Then the ample preordering gives $H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$. Since the auxiliary test functions are not given constructively, determining if a representation is contractive on these is in general difficult, but as mentioned above, n contractive representations will be completely contractive if n is sufficiently large. In the classical setting of Agler's realization theorem for the polydisk the auxiliary test functions are simply the test functions, and by our definition, the Brehmer representations are in this case just those representations mapping the coordinate functions to commuting contractions. Such representations are also completely contractive on $A(\mathbb{D}^d)$ with respect to the appropriate matrix norm structure (something which can also be gleaned from results in [37]). At first sight, this might seem paradoxical given Parrott's example of a commuting triple of contractions without a unitary dilation. However since the matrix norm structure is not that of H^∞ with the supremum norm, there is in fact no problem. Indeed, we show that there are choices in the Parrott example which give rise to a boundary representation (in the sense of Arveson), since it will be irreducible and not only will there be no commuting unitary dilation of the image of the coordinate functions, but in fact the only commuting contractive dilation is by means of a direct sum. Several other

matrix valued boundary representations are also explicitly given, one arising from an example of Grinshpan, Kaliuzhnyi-Verbovetskyi and Woerdeman [31], and another constructed from the Kaijser-Varopoulos example. It is also not difficult to see that the Crabbe-Davie example also gives a boundary representation. All of these send the coordinate functions to nilpotent matrices, though it can be shown that there must exist examples which are neither commuting unitaries nor unitarily equivalent to commuting nilpotents.

Finally, we demonstrate that in the setting of ample preorderings, Andô's theorem allows us to instead consider so-called nearly ample preorderings. With this we are able to recover the full extent of the results of Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman, and even generalize it, and at the same time improve the result mentioned in the previous paragraph by proving that when $d \geq 2$, for $n = 2^{d-2}$, n -contractive weakly continuous representations of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ and n -contractive representations of $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ are completely contractive. In particular, over the polydisk the images of the coordinate functions under such a representation will be commuting contractions with a commuting unitary dilation. Viewed another way, any example such as Parrot's of a representation of $A(\mathbb{D}^3)$ which is contractive but not completely contractive must fail to be 2-contractive.

2. TEST FUNCTIONS, PREORDERINGS, FUNCTION SPACES AND TOPOLOGY

2.1. Test functions and preorderings. Let X be a set, \mathcal{H} a Hilbert space, and Ψ a collection of $\mathcal{L}(\mathcal{H})$ valued functions on X . We call Ψ a set of **test functions** if for $x \in X$, $\sup_{\psi \in \Psi} \|\psi(x)\| < 1$, and when restricted to any finite set, Ψ generates all functions on that set (equivalently in the scalar valued case we are considering, Ψ separates the points of X). *We assume that the test functions we are dealing with are complex valued, though we later construct certain matrix valued test functions from these.*

There are many interesting situations where the collection of test functions is infinite [28]. *However, for this paper our focus will solely be on the situation when $d = |\Psi| := \text{card } \Psi$ is finite*, though many of the initial results are valid in any case. This assumption has the advantage of allowing us to, among other things, avoid additional complexities in the proof of the realization theorems, since when $|\Psi|$ is finite certain representations in which we will be interested have a particularly simple form.

We use standard tuple notation on $\bigoplus_1^d \mathbb{N}$, d -tuples of non-negative integers (n_i) , endowed with the partial ordering $n' \leq n$ if and only if $n'_i \leq n_i$ for all i . If $n = (n_i) \in \bigoplus_1^d \mathbb{N}$, we write $|n|$ for the sum of the entries of n . Also, we denote by e_i the tuple with all entries except the i th equal to zero, while the i th is 1, and 0 for the tuple where all entries are zero, and 1 will stand for the tuple with all entries equal to 1. We use the notation ψ^n to stand for $\prod \psi_i^{n_i}$, where the product is over the $n_i \in n$ which are nonzero.

By a **preordering** we mean a finite set $\Lambda \subset \bigoplus_1^d \mathbb{N}$ with the property that for all i , $e_i < \lambda$ for some $\lambda \in \Lambda$. This is at variance with the usual definition from real algebraic geometry, but happens to be more convenient in our context. The connection with the standard form should become apparent to those familiar with it.

We will see in the next section that for the applications we have in mind, the preordering is not unique, and in fact there are two rather special preorderings associated to any given preordering Λ . The first is the **minimal preordering** Λ_m , which is constructed from Λ as the union of all $\lambda \in \Lambda$ such that if $\lambda' \in \Lambda$ and $\lambda \leq \lambda'$, then $\lambda' = \lambda$. In other words, the minimal preordering consists of the union of the maximal elements Λ . The other is the **maximal preordering** $\Lambda_M := \{\lambda \in \bigoplus_{\psi \in \Psi} \mathbb{N} : \lambda \leq \lambda' \text{ for some } \lambda' \in \Lambda\}$. Hence if $\lambda' \in \Lambda$ and $\lambda \leq \lambda'$, then $\lambda \in \Lambda_M$.

We find it convenient to decompose any maximal preordering Λ_M as a disjoint union $\bigcup_{j=0}^{\infty} \Lambda_j$, where $\lambda \in \Lambda_j$ if and only if $|\lambda| = j$. Thus the only element of Λ_0 is 0, those in Λ_1 are the e_i s, and so on. Set Λ_+, Λ_- equal to the union over Λ_j s where the index is even and odd, respectively. Now for any $\lambda \in \Lambda_M$, there are $2^{|\lambda|}$ elements Λ_M which are less than or equal to λ , half of which are in Λ_+ and half in Λ_- . For the purposes of fixing a clear labeling on certain vectors later on, we use the ordering on Ψ to endow Λ_M with the lexicographic ordering \leq_ℓ .

Since $d = |\Psi| < \infty$, of particular interest will be the so-called **ample preorderings**. These are preorderings which have a largest element; that is, a unique maximal element, λ^m . When $\lambda^m = 1$, we call the resulting preordering a **standard ample preordering**. Thus if Λ is an ample preordering, the corresponding maximal preordering has the form $\Lambda := \{\lambda \in \mathbb{N}^d : \lambda \leq \lambda^m\}$. A minimal ample preordering thus consists of a single element, $\Lambda_m = \{\lambda^m\}$.

Let Λ be ample with maximal element λ^m , and $\lambda^1, \lambda^2 < \lambda^m$ where $\lambda > \lambda^1$ or λ^2 implies $\lambda = \lambda^m$. Then for $j = 1, 2$, $\lambda = \lambda^j + e_{\ell_j}$ for some e_{ℓ_j} , where the addition is entrywise. A preordering $\Lambda_s \subset \Lambda$ with the property that λ^1 and λ^2 are maximal elements in Λ_s is termed a **nearly ample preordering** under λ^m , and a **standard nearly ample preordering** when $\lambda^m = 1$.

A simple (indeed, unique) example of a standard nearly ample preordering when $d = |\Psi| = 2$ is $\lambda^1 = (1, 0)$, $\lambda^2 = (0, 1)$. For $d = 3$, there are three choices of minimal standard nearly ample preordering, such as for example, $\lambda^1 = (1, 1, 0)$, $\lambda^2 = (1, 0, 1)$.

2.2. Kernels and function spaces. Write $\mathcal{L}(\mathcal{H})$ for the bounded linear operators on a Hilbert space \mathcal{H} , $\mathcal{L}(\mathcal{H}, \mathcal{K})$ for the bounded linear operators mapping between Hilbert spaces \mathcal{H} and \mathcal{K} .

Let $\{\sigma_\lambda\}_{\lambda \in \Lambda}$ be a collection of $n_\lambda \times n_\lambda$ matrix valued functions on X such that for each x , $\sup_{\lambda \in \Lambda} \|\sigma_\lambda(x)\| < 1$. (These will later be the auxiliary test functions.) Define bounded functions E_x on Λ by $E_x(\lambda) = \sigma_\lambda(x)$. We use the notation $C_b(\Lambda)$ for the unital C^* -algebra generated by these functions. This is a finite dimensional algebra of dimension at most $\sum_\lambda n_\lambda$. As such, it is isomorphic to a direct sum of matrix algebras, and consequently, any representation will be (isomorphic to) a direct sum of identity representations applied to these matrix algebras. More specifically, for $\rho : C_b(\Lambda) \rightarrow \mathcal{L}(\mathcal{E})$, there will be orthogonal projections P_λ with orthogonal ranges such that $\mathcal{E} = \bigoplus_\lambda \text{ran } P_\lambda \otimes \mathbb{C}^{n_\lambda}$, and $\rho(E_x) = \bigoplus_\lambda P_\lambda \otimes \sigma_\lambda(x)$.

Let \mathcal{A} and \mathcal{B} be C^* -algebras. A kernel $\Gamma : X \times X \rightarrow \mathcal{L}(A, B)$ is called **completely positive** if for all finite sets $\{x_1, \dots, x_n\} \subset X$, $\{a_1, \dots, a_n\} \subset \mathcal{A}$ and $\{b_1, \dots, b_n\} \subset \mathcal{B}$,

$$\sum_{i,j=1}^n \left\langle \Gamma(x_i, x_j)(a_i a_j^*) b_i, b_j \right\rangle \geq 0.$$

A theorem due to Bhat, Barreto, Liebscher and Skeide [20, Theorem 3.6] shows that this is equivalent to the condition that for finite sets $\{x_1, \dots, x_n\} \subset X$, the matrix $(\Gamma(x_i, x_j))$ is a completely positive map from $M_n(\mathcal{A})$ to $M_n(\mathcal{B})$, and that this is further equivalent to the existence of a Kolmogorov decomposition for Γ . We state a special case of this suited to our purposes.

Proposition 2.1. *Let \mathcal{H} be a Hilbert space. The kernel $\Gamma : X \times X \rightarrow \mathcal{L}(C_b(\Lambda), \mathcal{L}(\mathcal{H}))$ is (completely) positive if and only if it has a **Kolmogorov decomposition**; that is, there exists a Hilbert space \mathcal{E} , a function $\gamma : X \rightarrow \mathcal{L}(\mathcal{E}, \mathcal{H})$ and a unital $*$ -representation $\rho : C_b(\Lambda) \rightarrow \mathcal{L}(\mathcal{E})$ such that*

$$\Gamma(x, y)(f g^*) = \gamma(x) \rho(f) \rho(g)^* \gamma(y)^*$$

for all $f, g \in C_b(\Lambda)$.

In the case of kernels $k : X \times X \rightarrow \mathcal{L}(\mathcal{H})$, which corresponds to replacing $C_b(\Lambda)$ by \mathbb{C} , it follows from standard results on completely positive maps, that positivity implies complete positivity. The existence of a Kolmogorov decomposition of positive operator valued kernels is originally due to Mlak [38]. We use the notation $\mathbb{K}_X^+(C_b(\Lambda), \mathcal{L}(\mathcal{H}))$ for the set of completely positive kernels on $X \times X$ with values in $\mathcal{L}(C_b(\Lambda), \mathcal{L}(\mathcal{H}))$.

For a fixed preordering Λ , the collection of kernels

$$\mathcal{K}_{\Lambda, \mathcal{H}} := \left\{ k : X \times X \rightarrow \mathcal{L}(\mathcal{H}) : k \geq 0 \text{ and for each } \lambda \in \Lambda, \right. \\ \left. \prod_{\lambda \ni \lambda_i \neq 0} ([1_{\mathcal{L}(\mathcal{H})}] - (\psi_i \otimes 1_{\mathcal{L}(\mathcal{H})})(\psi_i^* \otimes 1_{\mathcal{L}(\mathcal{H})}))^{\lambda_i} * k \geq 0 \right\},$$

are termed the **admissible kernels**. Here the kernel $[1_{\mathcal{L}(\mathcal{H})}]$ has all entries equal to $1_{\mathcal{L}(\mathcal{H})}$, the identity operator on \mathcal{H} , “ $*$ ” indicates the pointwise or Schur product of kernels, and $([1_{\mathcal{L}(\mathcal{H})}] - (\psi_i \otimes 1_{\mathcal{L}(\mathcal{H})})(\psi_i^* \otimes 1_{\mathcal{L}(\mathcal{H})}))^{\lambda_i}$ is the λ_i -fold Schur product of $[1_{\mathcal{L}(\mathcal{H})}] - (\psi_i \otimes 1_{\mathcal{L}(\mathcal{H})})(\psi_i \otimes 1_{\mathcal{L}(\mathcal{H})})^*$. In the non-scalar case we interpret this Schur product as follows: for a kernel F over $X \times X$,

$$([1_{\mathcal{L}(\mathcal{H})}] - (\psi_i \otimes 1_{\mathcal{L}(\mathcal{H})})(\psi_i \otimes 1_{\mathcal{L}(\mathcal{H})})^* * F)(x, y) := F(x, y) - (\psi_i(x) \otimes 1_{\mathcal{L}(\mathcal{H})})F(x, y)(\psi_i(y) \otimes 1_{\mathcal{L}(\mathcal{H})})^*.$$

More generally, if $F = f f^*$ and $G = g g^*$ are Kolmogorov decompositions of two positive kernels over Hilbert spaces \mathcal{F} and \mathcal{G} , respectively, then

$$F * G(x, y) = (f(x) \otimes g(x))(f(y) \otimes g(y))^*.$$

It is clear that the resulting kernel is positive.

The kernels in $\mathcal{K}_{\Lambda, \mathcal{H}}$ are then used to define the Banach algebra $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ consisting of those functions $\varphi : X \rightarrow \mathcal{L}(\mathcal{H})$ for which there is a finite constant $c \geq 0$ such that for all $k \in \mathcal{K}_{\Lambda, \mathcal{H}}$,

$$(c^2 [1_{\mathcal{L}(\mathcal{H})}] - \varphi \varphi^*) * k \geq 0,$$

and $\|\varphi\|$ is defined to be the smallest such c . We call the resulting algebra the **Agler algebra** and the norm the **Schur-Agler norm**. Denote the unit ball in this norm by $H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. This is referred to as the **Schur-Agler class**. In case the Agler algebra is isometrically isomorphic to $H^\infty(X)$, the unit ball is usually simply called the **Schur class**. It is not difficult to see that the function 1_X equaling $1_{\mathcal{L}(\mathcal{H})}$ at all x is in $H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ since $[1_{\mathcal{L}(\mathcal{H})}] = 1_X 1_X^*$. If $\mathcal{L}(\mathcal{H}) = \mathbb{C}$, we write $H^\infty(X, \mathcal{K}_\Lambda)$ and $H_1^\infty(X, \mathcal{K}_\Lambda)$ for $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ and $H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, respectively.

There are obvious modifications of these definitions which we will not explicitly state in the case of matrix valued test functions. For this setting, it will still be the case that the Agler algebra norm dominates the supremum norm.

For $\varphi \in H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, we can also define a norm by $\|\varphi\|_\infty := \sup_{x \in X} \|\varphi(x)\|$. This will in general be different from the norm defined above. Furthermore, since the kernel

$$k(y, z) = \begin{cases} 1 & y = z; \\ 0 & \text{otherwise,} \end{cases}$$

is admissible, it is apparent that $\|\varphi\|_\infty \leq \|\varphi\|$.

Two preorderings Λ_1 and Λ_2 are **equivalent preorderings** if for all Hilbert spaces \mathcal{H} , $\mathcal{K}_{\Lambda_1, \mathcal{H}} = \mathcal{K}_{\Lambda_2, \mathcal{H}}$, and consequently they generate the same Banach algebras.

Lemma 2.2. *Any preordering Λ is equivalent to both its minimal preordering Λ_m and its maximal preordering Λ_M .*

Proof. We prove the lemma when $\mathcal{H} = \mathbb{C}$, the other cases following in an identical manner.

It is clear that $\mathcal{K}_{\Lambda_M} \subseteq \mathcal{K}_{\Lambda} \subseteq \mathcal{K}_{\Lambda_m}$, so it suffices to ascertain that if $k \in \mathcal{K}_{\Lambda_m}$ and $\lambda \in \Lambda_M$, then

$$\prod_{\lambda \ni \lambda_i \neq 0} (1 - \psi_i \psi_i^*)^{\lambda_i} * k \geq 0.$$

Choose $\lambda' \in \Lambda_m$ such that $\lambda' \geq \lambda$. We may assume that $\lambda' \neq \lambda$, since otherwise there is nothing to show. Hence there is some i such that $p = \lambda'(i) - \lambda(i) > 0$. The kernel k_{ψ_i} with

$$k_{\psi_i}(x, y) = (1 - \psi_i(x) \psi_i(y)^*)^{-1} = \sum_n \psi_i^n(x) \psi_i^{n*}(y),$$

is positive on X . The Schur product of positive kernels is positive, so if we set $\tilde{\lambda} = \lambda' - p e_{\lambda_i}$ (the arithmetic done in the standard way), we find that

$$\prod_{\tilde{\lambda} \ni \lambda_i \neq 0} (1 - \psi_i \psi_i^*)^{\tilde{\lambda}_i} * k = k_{\psi_i}^p * \prod_{\tilde{\lambda} \ni \lambda_i \neq 0} (1 - \psi_i \psi_i^*)^{\tilde{\lambda}_i} * k \geq 0.$$

Continuing through those i such that $\lambda'(i) > \lambda(i)$, after a finite number of steps we achieve the desired result. \square

Lemma 2.3. *Given two preorderings Λ_1 and Λ_2 , if for the corresponding maximal preorderings $\Lambda_{1,M} \subseteq \Lambda_{2,M}$, then $\mathcal{K}_{\Lambda_2, \mathcal{H}} \subseteq \mathcal{K}_{\Lambda_1, \mathcal{H}}$. Consequently, $H^\infty(X, \mathcal{K}_{\Lambda_1, \mathcal{H}}) \subseteq H^\infty(X, \mathcal{K}_{\Lambda_2, \mathcal{H}})$, with the norm of any element in the first algebra greater than or equal to the value of the norm of that element in the second.*

Proof. If $\Lambda_{1,M} \subseteq \Lambda_{2,M}$, then every element $\lambda_1 \in \Lambda_{1,M}$ is less than or equal to a maximal element $\lambda_2^M \in \Lambda_{2,M}$, and so the first statement follows by arguing as in the last lemma. The other statements are immediate from the definitions of the admissible kernels and corresponding algebras. \square

We say that a kernel \tilde{k} is **subordinate** to another kernel k if there is a positive kernel F such that $\tilde{k} = k * F$. It is clear that if G is a difference of positive kernels such that $G * k \geq 0$ and \tilde{k} is subordinate to k , then $G * \tilde{k} \geq 0$. Hence if k is an admissible kernel, any kernel subordinate to k is also admissible.

The admissible kernels are particularly simple when we are dealing with standard ample pre-orderings, since they are all subordinate to a single kernel.

Lemma 2.4. *Let Λ be a standard ample preordering over $\Psi = \{\psi_1, \dots, \psi_d\}$. Then every kernel in $\mathcal{K}_{\Lambda, \mathcal{H}}$ is subordinate to*

$$k_s(x, y) := \left(1_{\mathcal{L}(\mathcal{H})} \otimes \prod_{j=1}^d (1 - \psi_j(x) \psi_j(y)^*)^{-1} \right).$$

Proof. Obviously k_s is an admissible kernel, since it is the inverse with respect to the Schur product of $\left(1_{\mathcal{L}(\mathcal{H})} \otimes \prod_{j=1}^d (1 - \psi_j(x) \psi_j(y)^*) \right)$. Hence if k is an admissible kernel, so that $\left(\prod_{j=1}^d (1 - (1_{\mathcal{L}(\mathcal{H})} \otimes \psi_j(x)) (1_{\mathcal{L}(\mathcal{H})} \otimes \psi_j(y)^*)) k(x, y) \right) = (F(x, y)) \geq 0$, then k is seen to be subordinate to k_s . \square

The lemma implies that when Λ is a standard ample preordering, it suffices to check membership in $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ against the single kernel k_s . There is an obvious version of this for ample preorderings as well, but since we will primarily be interested in the standard case, we do not state it.

Corollary 2.5. For $X = \mathbb{D}^d$ with $\Psi = \{z_1, \dots, z_d\}$ the coordinate functions and Λ the standard ample preordering, $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}}) = H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$, and all admissible kernels are subordinate to $k_s \otimes 1_{\mathcal{L}(\mathcal{H})}$, where k_s is the Szegő kernel

$$k_s(z, w) = \prod_{i=1}^d (1 - z_i w_i^*)^{-1}.$$

Proof. This follows from the observation that φ is in the unit ball of $H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$ if and only if $([1_{\mathcal{L}(\mathcal{H})}] - \varphi \varphi^*) * (k_s \otimes 1_{\mathcal{L}(\mathcal{H})}) \geq 0$, where $k_s = \prod_{j=1}^d (1 - z_j z_j^*)^{-1}$ is the Szegő kernel for the polydisk. \square

2.3. The realization theorem for the Schur class of the disk and Agler's generalization. The now classical realization theorem is an amalgam of various results, all characterizing the Schur class for the unit disk \mathbb{D} (that is the unit ball of $H^\infty(\mathbb{D})$). We state here the operator valued generalization (see, for example, [19]).

Theorem 2.6 (Classical Realization Theorem). *Let $\varphi : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{H})$. The following are equivalent:*

- (SC) $([1_{\mathcal{L}(\mathcal{H})}] - \varphi \varphi^*) * k_s \geq 0$, where $k_s(z, w) = (1 - z w^*)^{-1}$ is the Szegő kernel, or equivalently, $\varphi \in H_1^\infty(\mathbb{D}, \mathcal{L}(\mathcal{H}))$; that is, φ is in the Schur class;
- (AD) There is a positive kernel $\Gamma : \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{L}(\mathcal{H})$ such that $1_{\mathcal{L}(\mathcal{H})} - \varphi(z)\varphi(w)^* = \Gamma(z, w)(1 - z w^*)$;
- (TF) There is a Hilbert space \mathcal{E} and a unitary operator $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\mathcal{L}(\mathcal{E} \oplus \mathcal{H})$ such that

$$\varphi(z) = D + Cz(I - Az)^{-1}B;$$

- (vN) For every $T \in \mathcal{L}(\mathcal{H})$, \mathcal{H} a Hilbert space, with $\|T\| < 1$, $\|\varphi(T)\| \leq 1$.

The last item is a version of von Neumann's inequality. If $\varphi \in A(\mathbb{D}, \mathcal{L}(\mathcal{H}))$, the operator valued version of the disk algebra, then we may instead simply assume that $\|T\| \leq 1$ in von Neumann's inequality. We interpret $\varphi(T)$ as $D + (C \otimes T)(I - (A \otimes T))^{-1}B$. The third item is referred to as a transfer function representation, and (\mathcal{H}, U) is called a unitary colligation. The terminology comes from systems theory. The second item is called the Agler decomposition. In this case it is a trivial restatement of the first item. It becomes less trivial in the next theorem, which in the scalar version is due to Jim Agler [2] (see [19] for the operator valued case). We state it in terms of preorderings.

Theorem 2.7 (Agler's Realization Theorem for the polydisk). *Fix $d \in \mathbb{N}$, $\Lambda^o = \{e_j\}_{j=1}^d$ and let $\varphi : \mathbb{D}^d \rightarrow \mathcal{L}(\mathcal{H})$. The following are equivalent:*

- (SC) $([1_{\mathcal{L}(\mathcal{H})}] - \varphi \varphi^*) * k \geq 0$ for all $k \in \mathcal{K}_{\Lambda^o, \mathcal{H}}$, or equivalently $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$; that is, φ is in the Schur-Agler class;
- (AD) There are positive kernels $\Gamma_j : \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{L}(\mathcal{H})$, $j = 1, \dots, d$, such that $1_{\mathcal{L}(\mathcal{H})} - \varphi(z)\varphi(w)^* = \sum_j \Gamma_j(z, w)(1 - z_j w_j^*)$;
- (TF) There is a Hilbert space \mathcal{E} and a unitary operator $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\mathcal{L}(\mathcal{E} \oplus \mathcal{H})$ such that for $z \in \mathbb{D}^d$,

$$\varphi(z) = D + CZ(z)(I - AZ(z))^{-1}B,$$

where $Z(z) = \sum_j z_j P_j$ and $\sum_j P_j = 1_{\mathcal{L}(\mathcal{E})}$;

- (vN) For every d -tuple of commuting contractions $T = (T_1, \dots, T_d)$ with $T_j \in \mathcal{L}(\mathcal{H})$, \mathcal{H} a Hilbert space, with $\|T_j\| < 1$, we have $\|\varphi(T)\| \leq 1$.

We interpret $\varphi(T)$ in a similar manner as in the single variable case. As before, when $\varphi \in A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, we may instead simply assume $\|T_j\| \leq 1$ for all j . Various examples, including that of Kaijser and Varopoulos [46], show that when $d > 2$, the Schur-Agler class is a strict subset of the unit ball of $H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$. On the other hand, Andô's theorem implies that when $d = 2$, $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ and $H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$ coincide.

One of the most useful aspects of the realization theorem is that it allows us to do interpolation [19]. Since it particularly suits our needs, we state it in the setting of what is commonly known as tangential Nevanlinna-Pick interpolation.

Theorem 2.8. *Fix $d \in \mathbb{N}$, $\Lambda^o = \{e_j\}_{j=1}^d$ and domain \mathbb{D}^d . Let Ω be a subset of \mathbb{D}^d , and suppose that for Hilbert spaces \mathcal{H} and $\tilde{\mathcal{H}}$ there are functions $a, b : \Omega \rightarrow \mathcal{L}(\tilde{\mathcal{H}}, \mathcal{H})$ such that for any admissible kernel k for $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ restricted to Ω ,*

$$(aa^* - bb^*) * k \geq 0.$$

Then there is a function $\varphi \in H_1^\infty(\mathcal{K}_{\Lambda^o, \tilde{\mathcal{H}}})$ such that $b = \varphi a$, where the multiplication is pointwise.

The proof is essentially a reworking of the proof of the realization theorem, giving a function φ over Ω with a transfer function representation such that $b = \varphi a$. The transfer function representation immediately extends to all of \mathbb{D}^d , and so by the realization theorem, φ extends to a function in the unit ball of $H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^o, \tilde{\mathcal{H}}})$.

The identification of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ and $H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$ when $d = 2$ in the realization theorem uses a version of Andô's theorem [8] for $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ (the standard version of Andô's theorem corresponds to the case when $\mathcal{H} = \mathbb{C}$), as well as a theorem of Arveson's [10].

Theorem 2.9 (Andô's theorem). *Let $\pi : A(\mathbb{D}^2, \mathcal{L}(\mathcal{H})) \rightarrow \mathcal{L}(\mathcal{H})$ or $\pi : H^\infty(\mathbb{D}^2, \mathcal{L}(\mathcal{H})) \rightarrow \mathcal{L}(\mathcal{H})$ be a unital representation with the property that $\|\pi(1_{\mathcal{L}(\mathcal{H})} \otimes z_j)\| \leq 1$ or $\|\pi(1_{\mathcal{L}(\mathcal{H})} \otimes z_j)\| < 1$, respectively, for $j = 1, 2$. Then π dilates to a representation $\tilde{\pi}$ such that $\pi(1_{\mathcal{L}(\mathcal{H})} \otimes z_j)$ is unitary, and consequently, π is completely contractive.*

Proof. For any $F \in \mathcal{L}(\mathcal{H})$, the constant function $F \otimes 1$ is obviously in $A(\mathbb{D}^2, \mathcal{L}(\mathcal{H}))$. Thus π restricted to the constant functions induces a unital representation of $\mathcal{L}(\mathcal{H})$, and since $\mathcal{L}(\mathcal{H})$ is a C^* -algebra, the induced representation is contractive.

Now suppose that $\pi : A(\mathbb{D}^2, \mathcal{L}(\mathcal{H})) \rightarrow \mathcal{L}(\mathcal{H})$ with $\|\pi(1_{\mathcal{L}(\mathcal{H})} \otimes z_j)\| \leq 1$, $j = 1, 2$. Let $\{e_\alpha\}$ be an orthonormal basis for \mathcal{H} . Define an operator $\eta_{\alpha\beta} : \mathcal{L}(\mathcal{H}) \rightarrow \mathbb{C}$ by $\eta_{\alpha\beta}(F) := \langle Fe_\alpha, e_\beta \rangle$. Note that $\eta(F) := (\eta_{\alpha\beta}(F))_{\alpha\beta} = F$.

Let $\varphi \in A(\mathbb{D}^2, \mathcal{L}(\mathcal{H}))$ and define $\varphi_{\alpha\beta} \in A(\mathbb{D}^2)$ (the scalar valued bidisk algebra) by $\varphi_{\alpha\beta}(z) = \eta_{\alpha\beta}(\varphi(z))$. By the standard form of Andô's theorem, there is a pair of commuting unitary operators $U = (U_1, U_2)$ on $\mathcal{L}(\tilde{\mathcal{H}})$ such that for all α, β , $\varphi_{\alpha\beta}(T) = P_{\mathcal{H}} \varphi_{\alpha\beta}(U)|_{\mathcal{H}}$. Thus

$$\begin{aligned} \varphi(T) &= \eta \otimes 1_{\mathcal{L}(\mathcal{H})}(\varphi(T)) = (\varphi_{\alpha\beta}(T))_{\alpha\beta} \\ &= (P_{\mathcal{H}} \varphi_{\alpha\beta}(U)|_{\mathcal{H}})_{\alpha\beta} = (P_{\mathcal{H}}(\eta_{\alpha\beta} \otimes 1_{\mathcal{L}(\mathcal{H})})(\varphi(U))|_{\mathcal{H}})_{\alpha\beta} \\ &= P_{\mathcal{H} \otimes \mathcal{H}} \varphi(U)|_{\mathcal{H} \otimes \mathcal{H}}. \end{aligned}$$

The easy direction of a result of Arveson's [10] or the spectral mapping theorem and a bit of work then shows that π is completely contractive.

Next, suppose that $\pi : H^\infty(\mathbb{D}^2, \mathcal{L}(\mathcal{H})) \rightarrow \mathcal{L}(\mathcal{K})$ and $\|\pi(1_{\mathcal{L}(\mathcal{H})} \otimes z_j)\| < 1$, $j = 1, 2$. Let $n \in \mathbb{N}$. Then $H^\infty(\mathbb{D}^2, \mathcal{L}(\mathcal{H})) \otimes M_n(\mathbb{C}) = H^\infty(\mathbb{D}^2, \mathcal{L}(\mathcal{H}^n))$ and for $j = 1, 2$,

$$\|\pi^{(n)}(1_{\mathcal{L}(\mathcal{H}^n)} \otimes z_j)\| = \left\| \bigoplus_1^n \pi(1_{\mathcal{L}(\mathcal{H})}) \right\| = \|\pi(1_{\mathcal{L}(\mathcal{H})})\| < 1.$$

It then follows from Theorem 2.7 that $\pi^{(n)}$ is contractive, and so π is completely contractive. \square

Another fundamental theorem is due to Brehmer [22]. It states that a d -tuple of commuting contractions satisfying an extra positivity condition dilates to commuting unitary operators.

Theorem 2.10 (Brehmer's theorem). *Let $T = (T_1, \dots, T_d)$ be a d -tuple of commuting contractions on a Hilbert space \mathcal{H} satisfying*

$$\prod_1^d (1 - T_j T_j^*) \geq 0,$$

where the product is in the hereditary sense (that is, adjoints on the right). Then there is a Hilbert space $\tilde{\mathcal{H}}$ containing \mathcal{H} and a d -tuple of commuting unitaries $U = (U_1, \dots, U_d)$ such that for any polynomial p over \mathbb{C}^d , $p(T) = P_{\mathcal{H}} p(U)|_{\mathcal{H}}$.

Because polynomials in $p[\mathbb{C}^d] \otimes \mathcal{L}(\mathcal{H})$ are weakly dense in $H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$, the same reasoning as in the operator generalization of Andô's theorem, coupled with Arveson's result, gives an alternate version of Brehmer's theorem.

Theorem 2.11. *Let $\pi : A(\mathbb{D}^d, \mathcal{L}(\mathcal{H})) \rightarrow \mathcal{L}(\mathcal{K})$ or $\pi : H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H})) \rightarrow \mathcal{L}(\mathcal{K})$ be a unital representation with the property that*

$$\prod_{i=1}^d (1 - \pi(z_i \otimes 1_{\mathcal{L}(\mathcal{H})}) \pi(z_i \otimes 1_{\mathcal{L}(\mathcal{H})})^*) \geq 0,$$

where the product is hereditary (that is, adjoints on the right), is either positive or strictly positive, respectively. Also assume that either $\|\pi(1_{\mathcal{L}(\mathcal{H})} \otimes z_j)\| \leq 1$ or $\|\pi(1_{\mathcal{L}(\mathcal{H})} \otimes z_j)\| < 1$, respectively. Then π dilates to a representation $\tilde{\pi}$ with $\tilde{\pi}(1_{\mathcal{L}(\mathcal{H})} \otimes z_j)$ unitary, and as a consequence, π is completely contractive.

There is a version of Brehmer's theorem for standard nearly ample preorderings, but this requires further developments.

Notice that Agler's realization theorem (Theorem 2.7) with $d = 2$ and Corollary 2.5 tell us that, at least over the bidisk with the coordinate functions as test functions, the ample preordering and the (in this case, unique) nearly standard ample preordering are equivalent; that is, they generate the same algebra and norm. As it happens, we can extend this idea to the polydisk, and as we will see later, to more general sets of test functions and domains. This will be used to prove a generalization of the main result of [30] in Theorem 4.13.

Theorem 2.12. *Let Ψ be the set of coordinate functions over the polydisk \mathbb{D}^d , $d \geq 2$, Λ^a the standard ample preordering (so with largest element $\lambda^m = (1, \dots, 1)$), and Λ^{na} a standard nearly ample preordering under λ^m . Then Λ^a and Λ^{na} are equivalent preorderings.*

Proof. When $d = 2$ this has already been shown. Hence we assume $d > 2$.

Fix a nearly ample preordering Λ^{na} under $\lambda^m = (1, \dots, 1)$. Recall that by Corollary 2.5, with the standard ample preordering, all admissible kernels are subordinate to the Szegő kernel k_s . Since this kernel is also admissible for the standard nearly ample preordering, any $\varphi \in H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^{na}, \mathcal{H}})$

is in $H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^a, \mathcal{H}})$ and the norm in the first algebra $\|\varphi\|_{na}$ is greater than or equal to that in the second, $\|\varphi\|_a$.

We now show that the two norms are the same on $H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^{na}, \mathcal{H}})$. First of all, recall that Λ^{na} has two maximal elements λ_1^m and λ_2^m , which are λ^m with one of the 1s changed to a zero in distinct places. By relabeling if necessary, we may assume that these are in the first two places. Let k_3 be the positive kernel defined by $k_3(z, w) = 1_{\mathcal{L}(\mathcal{H})} \otimes \prod_{j=3}^d (1 - z_j w_j^*)^{-1}$. This has a Kolmogorov decomposition $k_3 = a a^*$, where $a : \mathbb{D}^d \rightarrow \mathcal{L}(\tilde{\mathcal{H}}, \mathcal{H})$. Any admissible kernel in $\mathcal{K}_{\Lambda^{na}}$ then has the form $k = k_3 * k'$, where for fixed z_3, \dots, z_d and $j = 1, 2$,

$$\left([1_{\mathcal{L}(\mathcal{H})}] \otimes (1 - z_j w_j^*) k_3(z, w) k'(z, w) \right) \geq 0.$$

This inequality is valid in particular for any positive kernel k' which is a function only of the first two coordinates such that for $j = 1, 2$, $\left([1_{\mathcal{L}(\mathcal{H})}] \otimes (1 - z_j w_j^*) k'(z, w) \right) \geq 0$; that is, by Theorems 2.7 and 2.9, the kernels for $H^\infty(\mathbb{D}^2, \mathcal{L}(\mathcal{H}))$.

Let $\varphi \in H_1^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^{na}, \mathcal{H}})$, and define $b : \mathbb{D}^d \rightarrow \mathcal{L}(\tilde{\mathcal{H}}, \mathcal{H})$ by the pointwise product $b = a \varphi$. Then for any admissible kernel $k = k_3 * k'$ as above for the standard nearly ample preordering, $([1_{\mathcal{L}(\mathcal{H})}] - \varphi \varphi^*) * k \geq 0$. For fixed z_3, \dots, z_d , $(a a^* - b b^*) * k' \geq 0$. By Theorem 2.8, we can also write $b = a \tilde{\varphi}$, where as a function of only the first two variables, $\tilde{\varphi} \in H_1^\infty(\mathbb{D}^2, \mathcal{L}(\mathcal{H}))$; that is, is analytic and has supremum norm less than or equal to 1.

Fix $z_1, z_2 \in \mathbb{D}$. Set the kernel k' to be

$$k'(z, w) = \begin{cases} 1_{\mathcal{L}(\mathcal{H})} & w_1 = z_1 \text{ and } w_2 = z_2; \\ 0 & \text{otherwise.} \end{cases}$$

Then $k = k_3 * k'$ is admissible for the standard nearly ample preordering, and so $([1_{\mathcal{L}(\mathcal{H})}] - \varphi \varphi^*) * k_3 \geq 0$. Hence for fixed z_1, z_2 , the function $\varphi \in H_1^\infty(\mathbb{D}^{d-2}, \mathcal{L}(\mathcal{H}))$; that is, as a function of z_3, \dots, z_d , φ is also analytic and bounded by 1. Being separately analytic in all variables and bounded, we conclude from Hartog's theorem that $\varphi \in H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$. Writing k_s^d for the scalar Szegő kernel on \mathbb{D}^d , it follows that

$$([1_{\mathcal{L}(\mathcal{H})}] - \varphi \varphi^*) * (1_{\mathcal{L}(\mathcal{H})} \otimes k_s^d) = (a a^* - b b^*) * (1_{\mathcal{L}(\mathcal{H})} \otimes k_s^2) = a a^* * ([1_{\mathcal{L}(\mathcal{H})}] - \tilde{\varphi} \tilde{\varphi}^*) * (1_{\mathcal{L}(\mathcal{H})} \otimes k_s^2) \geq 0;$$

that is, $\varphi \in H_1^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$. This shows that $H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^{na}, \mathcal{H}})$ is a norm closed subalgebra of $H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$ (with the same norm).

To finish the proof, we note from the transfer function representation for $H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^o, \mathcal{H}})$ from Theorem 2.7 that polynomials in the coordinate functions are weakly dense in $H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^o, \mathcal{H}})$. Since they are also weakly dense in $H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$ and by Lemma 2.3,

$$H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^o, \mathcal{H}}) \subseteq H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^{na}, \mathcal{H}}) \subseteq H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^a, \mathcal{H}}) = H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H})),$$

the result follows. \square

We finally mention an abstract version of the realization theorem for general domains and sets of test functions [28] which is also part of the inspiration for the work that follows. The theorem was only stated and proved for scalar valued functions, though the generalization to operator valued functions is straightforward, as we shall see.

Theorem 2.13 (Classical realization theorem). *Let X be a set, $\Psi = \{\psi_j\}$ a (not necessarily finite) collection of test functions on X , $\Lambda = \{e_j\}$, \mathcal{K} the set of $\mathcal{L}(\mathcal{H})$ valued admissible kernels, and $H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ the unit ball of the algebra generated by the kernels in \mathcal{K} in $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. Let $\varphi : X \rightarrow \mathcal{L}(\mathcal{H})$. The following are equivalent:*

- (SC) $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$;
- (AD) *There exists a positive kernel $\Gamma : C_0(\Psi) \rightarrow \mathcal{L}(\mathcal{H})$ so that $[1_{\mathcal{L}(\mathcal{H})}] - \varphi \varphi^* = \Gamma * ([1] - EE^*)$, where $E(x)(\psi_j) = \psi_j(x)$;*
- (TF) *There is a unitary colligation (U, \mathcal{E}, ρ) , where \mathcal{E} is a Hilbert space, $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ a unitary operator on $\mathcal{E} \oplus \mathcal{H}$ and $\rho : C_0(\Psi) \rightarrow \mathcal{L}(\mathcal{E})$ a unital representation such that φ has a transfer function representation*

$$\varphi(x) = D + CZ(x)(1_{\mathcal{L}(\mathcal{E})} - AZ(x))^{-1}B,$$

with $Z(x) = \rho(E)(x)$;

- (vN1) $\varphi \in H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ and for every unital representation π of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ satisfying $\|\pi(\psi_j)\| < 1$ for all j , we have $\|\pi(\varphi)\| \leq 1$.
- (vN2) $\varphi \in H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ and for every weakly continuous unital representation π of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ satisfying $\|\pi(\psi_j)\| \leq 1$ for all j , we have $\|\pi(\varphi)\| \leq 1$.

Note that the second item (the so-called Agler decomposition) has a more familiar form in this setting when the set of test functions is finite; namely, we can rewrite this as $[1_{\mathcal{L}(\mathcal{H})}] - \varphi \varphi^* = \sum_j \Gamma_j * ([1_{\mathcal{L}(\mathcal{H})}] - \psi_j \psi_j^*)$, where each Γ_j is a positive $\mathcal{L}(\mathcal{H})$ valued kernel, as in Theorem 2.7. A representation is **weakly continuous** if whenever (φ_α) is a bounded net in $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ converging pointwise in norm to $\varphi \in H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, $(\pi(\varphi_\alpha))$ converges weakly to $\pi(\varphi)$.

Recall that in this paper we are taking the set of test functions to be finite. Thus Z has the simpler form

$$Z(x) = \sum_j \psi_j(x) P_j,$$

where each P_j is an orthogonal projection and $\sum_j P_j = 1_{\mathcal{L}(\mathcal{E})}$.

2.4. Topologizing X . In the construction of kernels and function spaces, we did not assume that the underlying set X has a topology, though even when it does have one, it will be convenient to take it to have the weakest topology making the test functions continuous. In most cases of interest the test functions are already continuous when X has its native topology, so this assumption will make no substantial difference, at least when \bar{X} is compact in the original topology and the test functions extend continuously to \bar{X} .

Write $H^\infty(X, \mathcal{K}_\Lambda)^*_0$ for the vector space of continuous linear functionals on $H^\infty(X, \mathcal{K}_\Lambda)$. The set $\mathcal{N} = \{e \in H^\infty(X, \mathcal{K}_\Lambda)^*_0 : |e(\psi)| = 0 \text{ for all } \psi \in \Psi\}$ is a subspace, and we write $H^\infty(X, \mathcal{K}_\Lambda)^*$ for the quotient space $H^\infty(X, \mathcal{K}_\Lambda)^*_0 / \mathcal{N}$. The test functions induce a topology on $H^\infty(X, \mathcal{K}_\Lambda)^*$ with a subbase consisting of sets of the form

$$U_{w,c} = \{\eta \in H^\infty(X, \mathcal{K}_\Lambda)^* : \sup_{\psi \in \Psi} |\eta(\psi) - w| < c\}, \quad w \in \mathbb{D}, c \in (0, 1).$$

By construction the map $\hat{E} : X \rightarrow H^\infty(X, \mathcal{K}_\Lambda)^*$ define by $\hat{E}[x](\varphi) = \varphi(x)$ is an embedding by the point separation property of the test functions. With this topology $H^\infty(X, \mathcal{K}_\Lambda)^*$ becomes a locally compact, convex topological vector space, and so is Hausdorff (in fact it has even stronger separation properties, which we will not need). We identify the closure of X, \bar{X} with the closure of $\hat{E}[X]$ in $H^\infty(X, \mathcal{K}_\Lambda)^*$.

The space $H^\infty(X, \mathcal{K}_\Lambda)^*$ induces a weak-* topology on $H^\infty(X, \mathcal{K}_\Lambda)^{**}$ in which the norm closed unit ball of $H^\infty(X, \mathcal{K}_\Lambda)^{**}$ is compact by the Banach-Alaoglu theorem. By dint of being finite Ψ in $H^\infty(X, \mathcal{K}_\Lambda)^{**}$ is also compact.

We also find that the test functions extend continuously to \bar{X} . However it is not *a priori* evident that the test functions will separate the points of \bar{X} . However, this can be achieved simply by identifying those points in \bar{X} which are not distinguished by the test functions.

We could have carried out the same sort of construction replacing $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})^*$ by the space of bounded linear operators from $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ to $\mathcal{L}(\mathcal{H})$, once again modding out by those maps η with the property that $\eta(\Psi) = \{0\}$. Since any $\eta' \in H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})^*$ induces $\eta \in H^\infty(X, \mathcal{K}_\Lambda)^*$ by $\eta(\varphi) = \eta'(\varphi \otimes 1_{\mathcal{L}(\mathcal{H})})$, this essentially adds nothing new.

By construction, $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ is a norm closed subalgebra of $C_b(X, \mathcal{L}(\mathcal{H}))$, the C^* -algebra of bounded continuous $\mathcal{L}(\mathcal{H})$ valued functions on X ; that is, a subalgebra of $C(\beta X, \mathcal{L}(\mathcal{H}))$, where βX is the Stone-Ćech compactification of X . However \bar{X} generally tends to be quite a bit smaller than βX . For example, if $X = \mathbb{D}^d$ and the test functions are the coordinate functions, $\bar{X} = \overline{\mathbb{D}}^d$, as one would expect.

2.5. Continuity and convergence in $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ and $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. With X topologized as in the last subsection, we can now address the continuity of elements of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ (we show that they are all continuous) and connections between various topologies on $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ and $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, the subalgebra of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ which in analogy with the disk algebra, consists of those (continuous) elements of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ which extend continuously to \bar{X} . When dealing with $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ it is convenient to assume that the test functions are in this algebra.

Lemma 2.14. *Let Ψ be a finite set and $\varphi \in H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. Then φ is continuous; that is, if (x_α) is a net in X converging to $x \in X$, $\|\varphi(x_\alpha) - \varphi(x)\| \rightarrow 0$. Furthermore, given $\epsilon > 0$ and $x \in X$, there is an open set $U_x \ni x$ such that for all $\varphi \in H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, $\|\varphi(y) - \varphi(y')\| < \epsilon \|\varphi\|$ for all $y, y' \in U_x$.*

Proof. Fix $0 < \delta < 1$. By definition, for $x \in X$, $\sup_{\psi \in \Psi} |\psi(x)| = 1 - \tilde{\epsilon}$ for some $\tilde{\epsilon} > 0$. In fact, for any $\epsilon > 0$, it follows that since Ψ is a finite set, $U_{x, \epsilon} := \{y \in X : \sup_{\psi \in \Psi} |\psi(x) - \psi(y)| < \epsilon\}$ is a relatively open neighborhood of x .

We claim that for ϵ sufficiently small and $y \in U_{x, \epsilon}$, the kernel defined by

$$k_{x, y}(z, w) := \begin{cases} 1 & z = w = x \text{ or } z = w = y, \\ 1 - \delta & (z = x \text{ and } w = y) \text{ or } (z = y \text{ and } w = x), \\ 0 & \text{otherwise,} \end{cases}$$

is an admissible kernel for $H^\infty(X, \mathcal{K}_\Lambda)$, and hence $k_{x, y} \otimes 1_{\mathcal{L}(\mathcal{H})}$ is admissible for $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. We require that

$$\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \psi(x)\psi(x)^* & \psi(x)\psi(y)^* \\ \psi(y)\psi(x)^* & \psi(y)\psi(y)^* \end{pmatrix} \right)^* \begin{pmatrix} 1 & 1 - \delta \\ 1 - \delta & 1 \end{pmatrix} \geq 0.$$

This will be nonnegative as long as $(1 - \delta)^2 |1 - \psi(x)\psi(y)^*|^2 \leq (1 - \psi(x)\psi(x)^*)(1 - \psi(y)\psi(y)^*)$.

Since $\psi(x)\psi(y)^* = \psi(x)(\psi(x)^* - (\psi(x) - \psi(y))^*)$ and $|\psi(x)| < 1$,

$$\begin{aligned} |1 - \psi(x)\psi(y)^*|^2 &= |1 - \psi(x)\psi(x)^* + \psi(x)(\psi(x) - \psi(y))^*|^2 \\ &\leq (1 - \psi(x)\psi(x)^*)^2 + 2\epsilon(1 - \psi(x)\psi(x)^*) + \epsilon^2. \end{aligned}$$

Also,

$$1 - \psi(y)\psi(y)^* \geq (1 - \psi(x)\psi(x)^*) - 2\epsilon - \epsilon^2.$$

Hence it suffices to choose ϵ so that

$$(1 - \delta)^2 (c^2 + 2\epsilon c + \epsilon^2) \leq c^2 - 2\epsilon c - \epsilon^2 c,$$

where $c = 1 - \psi(x)\psi(x)^*$, or equivalently, so that

$$-(c + (1 - \delta)^2)\epsilon^2 - 2c(1 + (1 - \delta)^2)\epsilon + c^2(1 - (1 - \delta)^2) > 0.$$

This is a polynomial in ϵ which is positive when $\epsilon = 0$, and so by continuity is positive for sufficiently small $\epsilon > 0$. In fact it is positive on the interval $\epsilon \in (0, d)$, where

$$\begin{aligned} d &= \frac{-c(1 + (1 - \delta)^2) + \sqrt{c^2(1 + (1 - \delta)^2)^2 + c^2(c + (1 - \delta)^2)(1 - (1 - \delta)^2)}}{c + (1 - \delta)^2} \\ &\geq c \frac{-(1 + (1 - \delta)^2) + \sqrt{(1 + (1 - \delta)^2)^2 + (c + (1 - \delta)^2)(1 - (1 - \delta)^2)}}{1 + (1 - \delta)^2} \\ &= c \left(\sqrt{1 + \frac{(c + (1 - \delta)^2)(1 - (1 - \delta)^2)}{(1 + (1 - \delta)^2)^2}} - 1 \right) \\ &\geq \frac{c(c + (1 - \delta)^2)(1 - (1 - \delta)^2)}{2(1 + (1 - \delta)^2)^2} \\ &\geq c\delta \geq 2\tilde{\epsilon}\delta. \end{aligned}$$

Fix $\varphi \in H^\infty(X, \mathcal{H}_{\Lambda, \mathcal{H}})$. We may assume without loss of generality that $\|\varphi\| = 1$. Since $k_{x,y}$ is admissible,

$$\begin{pmatrix} 1 - \varphi(x)\varphi(x)^* & (1 - \delta)(1 - \varphi(x)\varphi(y)^*) \\ (1 - \delta)(1 - \varphi(y)\varphi(x)^*) & 1 - \varphi(y)\varphi(y)^* \end{pmatrix} \geq 0. \quad (2.1)$$

Let $h \in \mathcal{H}$ with $\|h\| = 1$, and set

$$\begin{aligned} c_1 &= \langle (1 - \varphi(x)\varphi(x)^*)h, h \rangle \geq 0, \\ c_2 &= \langle \varphi(x)(\varphi(x) - \varphi(y))^*h, h \rangle, \\ c_3 &= \|(\varphi(x) - \varphi(y))^*h\|^2. \end{aligned}$$

Then positivity of the matrix in (2.1) implies the scalar matrix

$$\begin{pmatrix} c_1 & (1 - \delta)(c_1 + c_2) \\ (1 - \delta)(c_1 + c_2^*) & c_1 + c_2 + c_2^* - c_3 \end{pmatrix} \geq 0, \quad (2.2)$$

which is equivalent to

$$(1 - (1 - \delta)^2)(c_1^2 + c_1 c_2^* + c_2 c_1) - (1 - \delta)^2 |c_2|^2 \geq c_1 c_3. \quad (2.3)$$

Since $1 \geq c_1$, $2 \geq |c_2|$ and $1 - (1 - \delta)^2 \geq 2\delta$, it follows from (2.3) that

$$10\delta - (1 - \delta)^2 |c_2|^2 \geq c_1 c_3,$$

and so $c_3 \leq 10\delta/c_1$ when $c_1 \neq 0$. So if $c_1 \geq \sqrt{\delta}$, then $c_3 \leq 10\sqrt{\delta}$. On the other hand, if $c_1 < \sqrt{\delta}$, then $\|\varphi(x)h\|^2 \geq 1 - \sqrt{\delta}$. Also, by (2.2)

$$0 \leq \begin{pmatrix} \sqrt{\delta} & (1 - \delta)(c_1 + c_2) \\ (1 - \delta)(c_1 + c_2^*) & \sqrt{\delta} + c_2 + c_2^* - c_3 \end{pmatrix} \leq \begin{pmatrix} \sqrt{\delta} & (1 - \delta)(c_1 + c_2) \\ (1 - \delta)(c_1 + c_2^*) & 5 \end{pmatrix}.$$

Consequently, $|c_1 + c_2| \leq 5\sqrt{\delta}/(1 - \delta)$, and so $|c_2| \leq 6\sqrt{\delta}/(1 - \delta)$. Then $\sqrt{\delta} + c_2 + c_2^* - c_3 \geq 0$ gives $c_3 \leq 13\sqrt{\delta}/(1 - \delta)$. Thus $\|\varphi(x) - \varphi(y)\| \leq 13\sqrt{\delta}/(1 - \delta)$ whenever $y \in U_{x,\epsilon}$. Note that by construction the set $U_{x,\epsilon}$ is independent of the choice of $\varphi \in H_1^\infty(X, \mathcal{H}_{\Lambda, \mathcal{H}})$.

Suppose (x_α) is a net converging to $x \in X$. We saw that given $\delta > 0$, there is an $\epsilon > 0$ such that $k_{x,y}$ is an admissible kernel. Also, there is a α_δ such that for all $\alpha > \alpha_\delta$, $x_\alpha \in U_{x,\epsilon}$. Hence by what we have shown, $\|\varphi(x) - \varphi(x_\alpha)\| \leq 13\sqrt{\delta}/(1-\delta)$. Since any open neighborhood of x contains a $U_{x,\epsilon}$ for sufficiently small ϵ , we conclude that φ is norm continuous. \square

Lemma 2.14 ensures that the definition of $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ makes sense, though of course at this point we do not know if it consists of any more than the constant functions on X . In the concrete examples most commonly considered, it is also the case that the test functions are in $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, and we will generally assume this to be the case, as well as that they separate the points of \bar{X} .

Lemma 2.15. *The space $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ is complete in the norm topology, and its norm closed unit ball $H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ is closed in both the topology of pointwise convergence and the topology of uniformly convergence on compact subsets of X .*

Proof. Let (φ_α) be a Cauchy net in $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. For fixed $x \in X$ let k_x be the kernel which equals $1_{\mathcal{L}(\mathcal{H})}$ in the (x, x) place and zero elsewhere. It is clear by definition of the test functions that this is an admissible kernel. By the assumption that (φ_α) is a Cauchy net, using the kernel k_x , we see that $(\varphi_\alpha(x))$ is a Cauchy net in $\mathcal{L}(\mathcal{H})$, and since this space is complete, $(\varphi_\alpha(x))$ converges in norm. We denote the limit by $\varphi(x)$.

We show that the function $\varphi : x \mapsto \varphi(x)$ is in $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. Given $\epsilon > 0$, there is an α_0 such that for all $\alpha, \beta > \alpha_0$ and any admissible kernel k , $(\epsilon[1_{\mathcal{L}(\mathcal{H})}] - (\varphi_\alpha - \varphi_\beta)(\varphi_\alpha - \varphi_\beta)^*) * k \geq 0$. From this we see that there is a constant $c > 0$ such that for all $\alpha > \alpha_0$, $\|\varphi_\alpha\| < c$.

Let $F \subset X$ be a finite set. Given $\epsilon > 0$, choose α_0 as above, and also so that for all $\alpha > \alpha_0$ and $x, y \in F$, $2c\|\varphi_\alpha(x) - \varphi(x)\| + \|\varphi_\alpha(y) - \varphi(y)\|^2 \leq \epsilon/|F|^2$. Letting I denote the kernel which is $1_{\mathcal{L}(\mathcal{H})}$ on the main diagonal and zero elsewhere, we have on $F \times F$,

$$\begin{aligned} & (c[1_{\mathcal{L}(\mathcal{H})}] + \epsilon I - \varphi\varphi^*) * k \\ &= (c[1_{\mathcal{L}(\mathcal{H})}] + \epsilon I - (\varphi_\alpha + (\varphi - \varphi_\alpha))(\varphi_\alpha^* + (\varphi - \varphi_\alpha)^*)) * k \\ &\geq (c[1_{\mathcal{L}(\mathcal{H})}] - \varphi_\alpha\varphi_\alpha^*) * k + (\epsilon - (2c\|\varphi - \varphi_\alpha\| - \|\varphi - \varphi_\alpha\|^2)I) * k \\ &\geq 0. \end{aligned}$$

Since F and ϵ are arbitrary, this shows that $(c[1_{\mathcal{L}(\mathcal{H})}] - \varphi\varphi^*) * k \geq 0$, and so $\varphi \in H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$.

Suppose that $\varphi_\alpha \rightarrow \varphi$ pointwise, where $\|\varphi_\alpha\| \leq 1$. Let k be any admissible kernel and F a finite subset of X . Given $\epsilon > 0$, there is some α_0 such that for all $\alpha > \alpha_0$ and all $x \in F$, $\|\varphi_\alpha(x) - \varphi(x)\| < \epsilon$. Then for $\kappa = \max_{x,y \in F} \|k(x, y)\|$,

$$\begin{aligned} & ((([1_{\mathcal{L}(\mathcal{H})}] + \varphi(x)\varphi(y)^*)k(x, y)) \\ &= (([1_{\mathcal{L}(\mathcal{H})}] + \varphi_\alpha(x)\varphi_\alpha(y)^*)k(x, y) \\ &\quad - 2(\varphi_\alpha(x)(\varphi(y) - \varphi_\alpha(y))^* + (\varphi(x) - \varphi_\alpha(x))\varphi_\alpha(y)^*)k(x, y) - (\varphi(x) - \varphi_\alpha(x))(\varphi(y) - \varphi_\alpha(y))^*k(x, y)) \\ &\geq -\epsilon\kappa(1 + \epsilon)I_{F \times F}, \end{aligned}$$

which goes to zero as we take ϵ to zero, showing that $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$.

Finally, if $\varphi_\alpha \rightarrow \varphi$ uniformly on compact subsets of $H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, then in particular it converges pointwise to φ , and hence by what we have shown, $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. \square

Lemma 2.16. *The algebra $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ is closed in the norm topology, the topology of uniform convergence, and the topology of pointwise convergence.*

Proof. Let $(\varphi_\alpha)_{\alpha \in A}$ be a net in $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ converging in norm in $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ to φ . We show that $\varphi \in A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$.

Let (x_β) be a net in X converging to $x \in \bar{X}$. By norm convergence, given any $\epsilon > 0$, there exists α_0 such that for all $\alpha_1, \alpha_2 > \alpha_0$ and all β , $\|\varphi_{\alpha_1}(x_\beta) - \varphi_{\alpha_2}(x_\beta)\| < \epsilon$. By continuity, $\|\varphi_{\alpha_1}(x) - \varphi_{\alpha_2}(x)\| < \epsilon$, and so $(\varphi_\alpha(x))$ is a Cauchy net and hence has a limit, which we denote by $\varphi(x)$. By continuity, $\varphi(x)$ is independent of the choice of net (x_β) .

We show that the function $\varphi : x \mapsto \varphi(x)$ is continuous on \bar{X} . Given $\epsilon > 0$, let V_x be an open ball in $\mathcal{L}(\mathcal{H})$ of radius $\epsilon/2$ about $\varphi(x)$, and set $\tilde{U}_x = \varphi^{-1}(V_x) \cap X$, an open set in X . Then let U_x be an open set in \bar{X} such that $U_x \cap X = \tilde{U}_x$ and note that $x \in U_x$. Let $y \in U_x$ and construct U_y in an identical manner. Obviously, $U_x \cap U_y \cap X \neq \emptyset$, and so if we choose w in this set, $\epsilon > \|\varphi(x) - \varphi(w)\| + \|\varphi(w) - \varphi(y)\| \geq \|\varphi(x) - \varphi(y)\|$. It follows that φ is continuous on \bar{X} .

The last two statements follow from the previous lemma. \square

2.6. Connections between $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ and algebras over the polydisk. Suppose either that X has a topology in which \bar{X} is compact (say for example, as a bounded subset of \mathbb{C}^d) or that X is endowed with a topology as in Subsection 2.4 which then ensures the continuity of the test functions and compactness of \bar{X} . In either case, we also assume that the test functions are in $A(X, \mathcal{K}_\Lambda)$ and that they separate the points of \bar{X} . Then there is a natural identification of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ and $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ with certain subalgebras of bounded analytic functions over subsets of the polydisk, which we give below.

Recall that by definition the test functions have the property that for $x \in X$,

$$z = \xi(x) := (\psi_1(x), \dots, \psi_d(x)) \in \mathbb{D}^d,$$

and that the test functions separate the points of X , or equivalently, that ξ is injective. By the assumption that the test functions are in $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ and that they separate the points of \bar{X} , which is compact, we have that $\xi(X) = \Omega \subseteq \mathbb{D}^d$, and $\xi(\bar{X}) = \bar{\Omega}$ is a compact subset of $\bar{\mathbb{D}}^d$.

Write $\Psi^{\text{pd}} = \{Z_1(z), \dots, Z_d(z)\}$, $z \in \mathbb{D}^d$, where $Z_j(z) = z_j$ are the coordinate functions, and take these as test functions over Ω . Let \mathcal{H} be a Hilbert space and write $\mathcal{K}_{\Lambda, \mathcal{H}}^{\text{pd}}$ for the admissible kernels in this setting, and $H_1^\infty(\mathcal{K}_{\Lambda, \mathcal{H}}^{\text{pd}})$ for the associated algebra with unit ball $H_1^\infty(\mathcal{K}_{\Lambda, \mathcal{H}}^{\text{pd}})$. In analogy with the Serre-Swan theorem, we have the following.

Lemma 2.17. *The map $\xi : \bar{X} \rightarrow \bar{\Omega} \subseteq \bar{\mathbb{D}}^d$ defined above is a homeomorphism. Consequently, given a preordering Λ and Hilbert space \mathcal{H} , there is an isometric unital algebra homomorphism from $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ onto $H^\infty(\Omega, \mathcal{K}_{\Lambda, \mathcal{H}}^{\text{pd}})$ and from $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ onto $A(\Omega, \mathcal{K}_{\Lambda, \mathcal{H}}^{\text{pd}})$.*

Proof. Since ξ is a bijection from \bar{X} to $\bar{\Omega}$, ξ^{-1} is well-defined, and so it suffices to show that ξ^{-1} is continuous. Suppose not. Then there is a net $(z_\alpha)_{\alpha \in A}$ converging to z in $\bar{\Omega}$ such that $x_\alpha = \xi^{-1}(z_\alpha)$ does not converge to $x = \xi^{-1}(z)$. Hence there is an open set U containing x with the property that for all α in A , there exists $\beta \geq \alpha$ such that $x_\beta \notin U$. The set $B = \{\beta \in A : x_\beta \notin U\}$ is thus a directed set. Since \bar{X} is compact, there is a subnet $(x_\gamma)_{\gamma \in \Gamma}$ of $(x_\beta)_{\beta \in B}$ converging to some $\tilde{x} \neq x$. But the subnet $(z_\gamma)_{\gamma \in \Gamma}$ converges to z , and so $\xi(\tilde{x}) = \xi(x)$, contradicting the injectivity of ξ .

It is then clear that for $k^{\text{pd}} \in \mathcal{K}_{\Lambda, \mathcal{H}}^{\text{pd}}$, k defined by $k(x, y) = k^{\text{pd}}(\xi(x), \xi(y))$ is in $\mathcal{K}_{\Lambda, \mathcal{H}}$, and similarly, that for $k \in \mathcal{K}_{\Lambda, \mathcal{H}}$, k^{pd} defined by $k^{\text{pd}}(z, w) = k(\xi^{-1}(z), \xi^{-1}(w))$ is in $\mathcal{K}_{\Lambda, \mathcal{H}}^{\text{pd}}$, giving a bijective correspondence between the sets of admissible kernels. It follows easily that $\nu : H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}}) \rightarrow H^\infty(\Omega, \mathcal{K}_{\Lambda, \mathcal{H}}^{\text{pd}})$ given by $\nu(\varphi)(z) = \varphi(\xi^{-1}(z))$ is an isometric unital algebra homomorphism and that $\nu(A(X, \mathcal{K}_{\Lambda, \mathcal{H}})) = A(\Omega, \mathcal{K}_{\Lambda, \mathcal{H}}^{\text{pd}})$. \square

Corollary 2.18. *Let Λ be an ample preordering and F a finite subset of X . Then the Szegő kernel restricted to $F \times F$ has closed range.*

Proof. Since the statement is true over the polydisk and the above map ξ is injective, the result is immediate. \square

2.7. Auxiliary test functions. Let $0 < \lambda \in \Lambda$. Define two $\mathbb{C}^{2^{|\lambda|-1}}$ valued functions by

$$\psi_{\lambda}^{+}(x) = \text{row}_{\lambda' \in \Lambda_{+}, \lambda' \leq_{\ell} \lambda} (\psi^{\lambda'}) \quad \text{and} \quad \psi_{\lambda}^{-}(x) = \text{row}_{\lambda' \in \Lambda_{-}, \lambda' \leq_{\ell} \lambda} (\psi^{\lambda'});$$

that is, ψ_{λ}^{+} has entries consisting of products of even numbers of ψ s (counting multiplicity) taken from ψ^{λ} , while ψ_{λ}^{-} has entries consisting of products of odd numbers of ψ s (counting multiplicity) taken from ψ^{λ} . We order these in increasing order, so that the first entry of ψ_{λ}^{+} is 1 (corresponding to $0 <_{\ell} \lambda$).

By construction, for $\lambda \in \Lambda$,

$$\prod_{\lambda_i \in \lambda} ([1] - \psi_i \psi_i^{*})^{\lambda_i}(x, y) = \psi_{\lambda}^{+}(x) \psi_{\lambda}^{+}(y)^{*} - \psi_{\lambda}^{-}(x) \psi_{\lambda}^{-}(y)^{*}$$

and for each $x \in X$,

$$\prod_{\lambda_i \in \lambda} ([1] - \psi_i \psi_i^{*})^{\lambda_i}(x, x) > 0.$$

From this we see that $|\psi_{\lambda}^{+}(x)|^2 = \psi_{\lambda}^{+}(x) \psi_{\lambda}^{+}(x)^{*} > 1$. For $\psi_{\lambda}^{+}(x) = |\psi_{\lambda}^{+}(x)| v_{\lambda}(x)$ the polar decomposition of $\psi_{\lambda}^{+}(x)$, we set

$$\omega_{\lambda}(x) := v_{\lambda}(x)^{*} |\psi_{\lambda}^{+}(x)|^{-1} = \psi_{\lambda}^{+}(x)^{*} |\psi_{\lambda}^{+}(x)|^{-2}.$$

Then $\psi_{\lambda}^{+}(x) \omega_{\lambda}(x) = 1$. Note that $\|\omega_{\lambda}(x)\| = |\psi_{\lambda}^{+}(x)|^{-1} < 1$. Define

$$\sigma_{\lambda}(x) := \omega_{\lambda}(x) \psi_{\lambda}^{-}(x) = \psi_{\lambda}^{+}(x)^{*} |\psi_{\lambda}^{+}(x)|^{-2} \psi_{\lambda}^{-}(x) \in M_{2^{|\lambda|-1}}(\mathbb{C}).$$

Obviously for all $x \in X$,

$$\psi_{\lambda}^{+}(x) \sigma_{\lambda}(x) = \psi_{\lambda}^{-}(x) \quad \text{and} \quad \|\sigma_{\lambda}(x)\| < 1.$$

As defined, $\sigma_{\lambda}(x)(\text{ran } \psi_{\lambda}^{-*}(x)) \subseteq \psi_{\lambda}^{+*}(x)$ and $\sigma_{\lambda}(x)(\ker \psi_{\lambda}^{-}(x)) = \{0\}$. Additionally, if the test functions are in $A(X, \mathcal{K}_{\Lambda})$, then $\sigma_{\lambda} \in C(\bar{X}, M_n(\mathbb{C}))$ where $n = 2^{|\lambda|-1}$.

Let $n = 2^{|\lambda|-1}$ and 1_n be the identity matrix on \mathbb{C}^n . Then

$$\begin{aligned} & (\psi_{\lambda}^{+} \psi_{\lambda}^{+*} * ([1_n] - \sigma_{\lambda} \sigma_{\lambda}^{*})) * (k \otimes 1_n)(x, y) \\ &= \psi_{\lambda}^{+}(x) ((k(x, y) \otimes 1_n) - \sigma_{\lambda}(x) (k(x, y) \otimes 1_n) \sigma_{\lambda}(y)^{*}) \psi_{\lambda}^{+}(y)^{*} \\ &= \psi_{\lambda}^{+}(x) (k(x, y) \otimes 1_n) \psi_{\lambda}^{+}(y)^{*} - \psi_{\lambda}^{-}(x) (k(x, y) \otimes 1_n) \psi_{\lambda}^{-}(y)^{*} \\ &= \left(\prod_{\lambda_i \in \lambda} ([1] - \psi_i \psi_i^{*})^{\lambda_i} * k \right) (x, y). \end{aligned} \tag{2.4}$$

We call the functions σ_{λ} , $\lambda \in \Lambda$ **auxiliary test functions**. The last calculation shows that we apparently only have positivity of $([1_n] - \sigma_{\lambda} \sigma_{\lambda}^{*}) * (k \otimes 1_n)$ after taking the Schur product with $\psi_{\lambda}^{+} \psi_{\lambda}^{+*}$, though clearly $\sigma_{\lambda} \in H^{\infty}(X, \mathcal{K}_{\Lambda})$ if $\lambda = e_j$ for some j , since when $|\lambda| = 1$, the auxiliary test functions are just the ordinary test functions. We examine this point more closely in the next section.

Fixing $x, y \in X$, we use the above to construct certain continuous functions over Λ . In particular, define

$$\begin{aligned} E^{\pm}(x)(\lambda) &= \psi_{\lambda}^{\pm}(x) \\ D(x, y)(\lambda) &= \prod_{j=1}^d (1 - \psi_j(x) \psi_j(y)^{*})^{\lambda_j}. \end{aligned}$$

Then

$$E^{+}(x)(\lambda) E^{+}(y)(\lambda)^{*} - E^{-}(x)(\lambda) E^{-}(y)(\lambda)^{*} = D(x, y)(\lambda), \tag{2.5}$$

and $E^+(x)(\lambda)E^+(x)(\lambda)^* \geq 1$.

2.8. Auxiliary test functions for ample preorderings. We show in this subsection that when Λ is an ample preordering, the auxiliary test function σ_λ can be modified so as to obtain a matrix valued $H^\infty(X, \mathcal{K}_\Lambda)$ function.

Theorem 2.19. *Assume that Ψ is a finite collection of test functions over a set X , Λ an ample preordering with maximal element λ^m . Then for $\lambda \in \Lambda$, $n = 2^{|\lambda|-1}$, the auxiliary test function σ_λ can be defined so that $([1_n] - \sigma_\lambda \sigma_\lambda^*) * (k_s \otimes 1_n) \geq 0$ and for all $x \in X$, $\|\sigma_\lambda(x)\| < 1$; that is, $\sigma_\lambda \in H_1^\infty(\mathcal{K}_{\Lambda, \mathbb{C}^n})$. If the test functions are in $A(X, \mathcal{K}_\Lambda)$, then we have $\sigma_\lambda \in A(\mathcal{K}_{\Lambda, \mathbb{C}^n})$. Furthermore, $k \in \mathcal{K}_{\Lambda, \mathcal{H}}$ if and only if $([1_n] - \sigma_{\lambda^m} \sigma_{\lambda^m}^*) * (k \otimes 1_n) \geq 0$, $n = 2^{|\lambda^m|-1}$. As a consequence, if we define a positive kernel k_Λ by $k_\Lambda(x, y) = (1 - \sigma_{\lambda^m}(x) \sigma_{\lambda^m}(y)^*)^{-1}$, then $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ if and only if $([1_n \otimes 1_{\mathcal{L}(\mathcal{H})}] - (1_n \otimes \varphi)(1_n \otimes \varphi)^*) * (k_\Lambda \otimes 1_{\mathcal{L}(\mathcal{H})}) \geq 0$.*

Proof. Fix $\lambda \in \Lambda$ and let F be a finite subset of X . Let $n = 2^{|\lambda|-1}$. We have a Kolmogorov decomposition of the Szegő kernel

$$k_s(x, y) = 1_n \otimes \prod_{j=1}^d (1 - \psi_j(x) \psi_j(y)^*)^{-1} = \kappa_s(x) \kappa_s(y)^*,$$

where $\kappa_s(x) : \mathcal{E} \rightarrow \mathbb{C}^n$, and \mathcal{E} is a Hilbert space equal to the closed span over $x \in X$ of the functions $\kappa_s(x)^*$.

Write k_F for the restriction of k to $F \times F$ and let $\mathcal{E}_F = \bigvee_{x \in F} \kappa_s(x)^* \subset \mathcal{E}$. By Corollary 2.18, k_F has closed range isomorphic to $\mathbb{C}^{n|F|}$. Denote by κ_F the column over $x \in F$ of $k_s(x)$ restricted to \mathcal{E}_F . Then $\text{ran } \kappa_F^* = \mathcal{E}_F$ and $\kappa_F \kappa_F^* = k_F$. Let Ψ_F^+ be a matrix with diagonal elements $\psi_\lambda^+(x)$, which maps from the range of κ_F , and let P_F^+ be the orthogonal projection onto the range of Ψ_F^{+*} . Also write σ_F for the direct sum over $x \in F$ of $\sigma_\lambda(x)$. Define

$$Q_F = \kappa_F^{-1} P_F^+ \kappa_F \quad \text{and} \quad G_F^o = P_{\text{ran } Q_F^*} \kappa_F^{-1} \sigma_F \kappa_F.$$

Here $P_{\text{ran } Q_F^*}$ is the orthogonal projection onto the range of Q_F^* .

It is clear that Q_F is a (not necessarily orthogonal) projection. Also, $\Psi_F^+ \kappa_F = \Psi_F^+ P_F^+ \kappa_F = \Psi_F^+ \kappa_F Q_F$ and $\kappa_F G_F^o = \sigma_F \kappa_F = P_F^+ \sigma_F \kappa_F = \kappa_F Q_F G_F^o$. Now,

$$\begin{aligned} 0 &\leq \Psi_F^+ (k_F - \sigma_F k_F \sigma_F^*) \Psi_F^{+*} \\ &= \Psi_F^+ \kappa_F Q_F (1 - G_F^o G_F^{o*}) \kappa_F^* Q_F^* \Psi_F^{+*}, \end{aligned}$$

and so $P_{\text{ran } Q_F^*} - G_F^o G_F^{o*} \geq 0$. Consequently, $\|G_F^o\| \leq 1$.

Let

$$\mathcal{G}_F = \{G_F \in \mathcal{L}(\mathcal{E}_F) : \|G_F\| \leq 1 \text{ and } P_{\text{ran } Q_F^*} G_F = G_F^o\}.$$

This is a compact subset of the unit ball of $\mathcal{L}(\mathcal{E}_F)$, and for any $G_F \in \mathcal{G}_F$,

$$\begin{aligned} \Psi_F^+ \kappa_F (1 - G_F G_F^*) \kappa_F^* \Psi_F^{+*} &= \Psi_F^+ \kappa_F Q_F (1 - G_F G_F^*) \kappa_F^* Q_F^* \Psi_F^{+*} \\ &= \Psi_F^+ (k_F - \sigma_F k_F \sigma_F^*) \Psi_F^{+*} \geq 0. \end{aligned}$$

If we define

$$\mathcal{S}_F = \{S_F \in \mathcal{L}(\text{ran } k_F) : S_F = \kappa_F G_F \kappa_F^{-1} \text{ for some } G_F \in \mathcal{G}_F\},$$

then by construction this set is nonempty, $k_F - S_F k_F S_F^* = \kappa_F (1 - G_F G_F^*) \kappa_F^* \geq 0$, and

$$\begin{aligned} \Psi_F^+ (k_F - S_F k_F S_F^*) \Psi_F^{+*} &= \Psi_F^+ (k_F - \sigma_F k_F \sigma_F^*) \Psi_F^{+*} \\ &= ((\psi_\lambda^+ \psi_\lambda^{+*} * ([1_n] - \sigma_\lambda \sigma_\lambda^*) * k_s)(x, y))_{x, y \in F}. \end{aligned} \tag{2.6}$$

The set \mathcal{S}_F is isomorphic to \mathcal{G}_F , and so is also compact. It is natural to define the norm of $S_F \in \mathcal{S}_F$ to equal the norm of the associated $G_F \in \mathcal{G}_F$. If $F' \supset F$ and $G_{F'} \in \mathcal{G}_{F'}$, then since κ_F is the restriction of $\kappa_{F'}$ to \mathcal{E}_F , $G_F := P_{\mathcal{E}_F} G_{F'} \in \mathcal{G}_F$. Thus for $S_{F'} = \kappa_{F'} G_{F'} \kappa_{F'}^{-1}$ and $S_F = \kappa_F G_F \kappa_F^{-1}$, the map $\pi_F^{F'} : \mathcal{S}_{F'} \rightarrow \mathcal{S}_F$ defined by

$$\pi_F^{F'}(S_{F'}) = S_F$$

is contractive and so continuous. Let \mathcal{F} be the collection of all finite subsets of X partially ordered by inclusion. The triple $(\mathcal{S}_F, \pi_F^{F'}, \mathcal{F})$ is an inverse limit of nonempty compact spaces, and so by Kurosh's Theorem [4, p. 30], for each $F \in \mathcal{F}$ there is an $S_F \in \mathcal{S}_F$ so that whenever $F, F' \in \mathcal{F}$ and $F \subset F'$,

$$\pi_F^{F'}((S_{F'})) = (S_F).$$

We can thus define $\tilde{\sigma}_\lambda$ on X by

$$\tilde{\sigma}_\lambda(x) = S_{\{x\}}.$$

By construction,

$$([1_n] - \tilde{\sigma}_\lambda \tilde{\sigma}_\lambda^*) * k_s \geq 0; \quad (2.7)$$

that is, $\tilde{\sigma}_\lambda \in H^\infty(X, \mathcal{K}_{\Lambda, \mathbb{C}^n})$.

It follows in particular from (2.7) that for any kernel k subordinate to the Szegő kernel k_s and $n = 2^{|\lambda^m|-1}$,

$$([1_n] - \tilde{\sigma}_{\lambda^m} \tilde{\sigma}_{\lambda^m}^*) * (k \otimes 1_n) \geq 0. \quad (2.8)$$

Also, by (2.6) any kernel k for which (2.8) holds is subordinate to the Szegő kernel. Hence the collection of auxiliary test functions constructed gives the same set of admissible kernels, and so generates $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ with the same norm.

Let $\lambda = \lambda^m$, with $\tilde{\sigma}_\lambda$ constructed as above. For the time being, we assume $\Lambda = \{\lambda^m\}$. Suppose that $x \in X$ has the property that $\|\tilde{\sigma}_\lambda(x)\| = 1$. Since $\tilde{\sigma}_\lambda(x) \in M_n(\mathbb{C})$ for $n = 2^{|\lambda|-1} < \infty$, there is some $f \in \mathbb{C}^n$ such that $\|\tilde{\sigma}_\lambda(x)f\| = \|f\| = 1$.

The test functions all have absolute value less than one in X , so for $y \neq x \in X$, the Szegő kernel satisfies $k_s(x, y) \neq 0$ and $k_s(x, x) > 0$, and by Corollary 2.18 when restricted to the two point set $\{x, y\} \subset X$, k_s is invertible. Consequently, $k_s(x, y) = k_s(x, x)^{1/2} g k_s(y, y)^{1/2}$, where $|g| < 1$.

Let $k_s(x, y) = \langle k_x, k_y \rangle$ be the Kolmogorov decomposition of k_s . Since (2.7) holds over the set $\{x, y\}$ and

$$\langle f \otimes k_x, f \otimes k_x \rangle - \langle \tilde{\sigma}_\lambda(x)f \otimes k_x, \tilde{\sigma}_\lambda(x)f \otimes k_x \rangle = 0,$$

it follows that

$$\langle f \otimes k_x, f \otimes k_y \rangle - \langle \tilde{\sigma}_\lambda(x)f \otimes k_x, \tilde{\sigma}_\lambda(y)f \otimes k_y \rangle = 0.$$

Hence there is an isometry V such that $\tilde{\sigma}_\lambda(x)f = \tilde{\sigma}_\lambda(y)f = Vf$.

Define the $M_n(\mathbb{C})$ valued kernel $\tilde{k}(z, w)$ to be equal to P_f , the projection onto the span of f if $z, w \in \{x, y\}$, and 0 otherwise. Since $([1_n] - \tilde{\sigma}_\lambda \tilde{\sigma}_\lambda^*) * \tilde{k} = 0$, \tilde{k} is admissible. On the other hand, to be admissible, it must also be the case that there is some positive kernel F such that $\tilde{k} = k_s * F$. Obviously, $F(z, w)$ must be zero if $z, w \notin \{x, y\}$ and $F(z, z) = k_s(z, z)^{-1} \otimes P_f$ for $z = x$ or y . Positivity then implies that $F(x, y) = (k_s(x, x)^{-1} g' k_s(y, y)^{-1}) \otimes P_f$ with $|g'| \leq 1$. Hence $(k_s * F)(x, y) = g g' P_f \neq P_f$, giving a contradiction. We conclude that for all x , $\|\tilde{\sigma}_{\lambda^m}(x)\| < 1$.

Now suppose that Λ is any ample preordering. If $\lambda \in \Lambda$, $\lambda \neq \lambda^m$, has the property that at some x , $\|\tilde{\sigma}_\lambda(x)\| = 1$, then an identical argument shows that the norm is achieved on a subspace \mathcal{F} , and that there is an isometry V such that for all $f \in \mathcal{F}$ and all $y \in X$, $\tilde{\sigma}_\lambda(y)f = Vf$. Consequently, we can change $\tilde{\sigma}_\lambda$ so that $\tilde{\sigma}_\lambda(y)f = 0$ for all $y \in X$. Testing against the Szegő kernel, it is clear that the resulting function is still in $H^\infty(X, \mathcal{K}_{\Lambda, \mathbb{C}^n})$ for appropriate n and now satisfies $\|\tilde{\sigma}_\lambda(x)\| < 1$. \square

Corollary 2.20. *Let $d \in \mathbb{N}$ and $n = 2^{d-1}$. There is a function $\sigma \in H_1^\infty(\mathbb{D}^d, M_n(\mathbb{C}))$ such that the set of positive kernels \mathcal{K} with the property that $k \in \mathcal{K}$ if and only if $([1_n] - \sigma\sigma^*) * k \geq 0$ are all subordinate to $1_n \otimes k_s$, where k_s is the the Szegő kernel $\prod_{j=1}^d (1 - z_j z_j^*)^{-1}$, and so $H^\infty(\mathbb{D}^d, \mathcal{K}_{\sigma, \mathcal{K}}) = H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$.*

Proof. This is a consequence of the last theorem and 2.5. □

2.9. Representations of $C_b(\Lambda)$. As noted previously, since $|\Lambda| < \infty$, a unital representation $\rho : C_b(\Lambda) \rightarrow \mathcal{L}(\mathcal{E})$, \mathcal{E} a Hilbert space, will have the form

$$\rho(f) = \sum_{\lambda \in \Lambda} P_\lambda \otimes f(\lambda),$$

where the P_λ s are orthogonal projections with orthogonal ranges and $\sum_{\lambda \in \Lambda} \text{ran } P_\lambda \otimes \mathbb{C}^{2|\lambda|-1} = \mathcal{E}$. We then naturally define

$$Z^\pm(x) := \sum_{\lambda \in \Lambda} P_\lambda \otimes E^\pm(x)(\lambda) = \sum_{\lambda \in \Lambda} P_\lambda \otimes \psi_\lambda^\pm(x)$$

and

$$R(x, y) = \rho(D(x, y)) := \sum_{\lambda \in \Lambda} P_\lambda \otimes D(x, y)(\lambda).$$

It follows that

$$Z^+(x)Z^+(y)^* - Z^-(x)Z^-(y)^* = R(x, y),$$

and $Z^+(x)Z^+(x)^* \geq 1$. In particular, $Z^+(x)Z^+(x)^*$ is invertible.

The operator $Z^+(x)$ has a right inverse given by

$$Y(x) := \sum_{\lambda \in \Lambda} P_\lambda \otimes \omega_\lambda(x) = \sum_{\lambda \in \Lambda} P_\lambda \otimes \psi_\lambda^+(x)^* |\psi_\lambda^+(x)|^{-2},$$

and so $P(x) = Y(x)Z^+(x)$ is the orthogonal projection onto $\overline{\text{ran}} Z^+(x)^*$.

Setting

$$S(x) := Y(x)Z^-(x) = \sum_{\lambda \in \Lambda} P_\lambda \otimes \sigma_\lambda(x),$$

we have $Z^-(x) = Z^+(x)S(x)$. Also, since $P(x)Y(x) = Y(x)Z^+(x)Y(x) = Y(x)$, we have $P(x)S(x) = S(x)$. Thus

$$\begin{aligned} 1 - S(x)S(x)^* &= 1 - Y(x)Z^-(x)Z^-(x)^*Y(x)^* \\ &= 1 - P(x)P(x) + P(x)P(x) - Y(x)Z^-(x)Z^-(x)^*Y(x)^* \\ &= 1 - P(x)P(x) + Y(x)Z^+(x)Z^+(x)^*Y(x)^* - Y(x)Z^-(x)Z^-(x)^*Y(x)^* \\ &= 1 - P(x)P(x) + Y(x) \left(Z^+(x)Z^+(x)^* - Z^-(x)Z^-(x)^* \right) Y(x)^* > 0. \end{aligned}$$

In case the preordering is ample, in the definition of $S(x)$ we may use Theorem 2.19 to replace σ_λ by a corresponding element of $H^\infty(X, \mathcal{K}_{\Lambda, \mathbb{C}^{2|\lambda|-1}})$.

We summarize in the following lemma.

Lemma 2.21. *Let $x \in X$.*

- (i) $Z^+(x)Z^+(x)^* - Z^-(x)Z^-(x)^* \geq 0$.
- (ii) *The operator $Y(x)$ is a right inverse of $Z^+(x)$ and $P(x) := Y(x)Z^+(x)$ is the orthogonal projection onto $\text{ran} Z^+(x)^*$.*
- (iii) *The operator $S(x) = Y(x)Z^-(x) : \overline{\text{ran} Z^-(x)^*} \rightarrow \text{ran} Z^+(x)^*$ (or a corresponding element of $H^\infty(X, \mathcal{K}_{\Lambda, \mathbb{C}^{2|\lambda|-1}})$ in case of an ample preordering) satisfies $Z^-(x) = Z^+(x)S(x)$ and has the property that $\|S(x)\| < 1$.*

Although it has not been explicitly indicated, it is worth bearing in mind that Z^+ , Z^- , S and so on, depend both on Λ and the choice of representation, and we will at times make this dependence explicit by writing $Z_{\Lambda, \rho}^+$, $Z_{\Lambda, \rho}^-$, $S_{\Lambda, \rho}$, etc.

3. TRANSFER FUNCTIONS, BREHMER REPRESENTATIONS AND DILATIONS

3.1. The transfer function algebra. In the standard manner, we define a $C_b(\Lambda)$ -**unitary colligation** Σ as a triple (U, \mathcal{E}, ρ) , where \mathcal{E} is a Hilbert space, $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{B}(\mathcal{E} \oplus \mathcal{H})$ is a unitary operator, and $\rho : C_b(\Lambda) \rightarrow \mathcal{L}(\mathcal{E})$ a unital $*$ -representation.

Definition 3.1. Assume the notation from Lemma 2.21. Given a $C_b(\Lambda)$ -unitary colligation Σ , we define the **transfer function** $W_\Sigma : X \rightarrow \mathcal{L}(\mathcal{H})$ associated to Σ as

$$W_\Sigma(x) := D + CS(x)(1 - AS(x))^{-1}B,$$

where $S = S_{\Lambda, \rho}$. Write

$$\mathcal{T}_1(X, \Lambda, \mathcal{H}) := \{W_\Sigma : \Sigma \text{ a unitary colligation}\},$$

and $\mathcal{T}(X, \Lambda, \mathcal{H})$ for the scalar multiples of elements in $\mathcal{T}_1(X, \Lambda, \mathcal{H})$. It is clear that $W \in \mathcal{T}(X, \Lambda, \mathcal{H})$ will not in general be uniquely represented as a multiple of a single element of $\mathcal{T}_1(X, \Lambda, \mathcal{H})$. For $W \in \mathcal{T}(X, \Lambda, \mathcal{H})$, define a norm by

$$\|W\| := \inf \{c \geq 0 : W = c W_\Sigma \text{ for some unitary colligation } \Sigma\}. \quad (3.1)$$

(We show in Theorem 3.3 below that $\|\cdot\|$ on $\mathcal{T}(X, \Lambda, \mathcal{H})$ really is a norm.) Finally, we write $\mathcal{T}^A(X, \Lambda, \mathcal{H})$ for the set of those $W \in \mathcal{T}(X, \Lambda, \mathcal{H})$ which extend continuously to \bar{X}

The formula gives the standard form of the transfer function when $\Lambda = \Lambda_1 = \{e_\psi\}$. Again, one should bear in mind that S depends on ρ .

More generally, we might also consider $C_b(\Lambda)$ -**contractive colligations** by allowing U to be contractive rather than unitary, and then likewise define a transfer function. As it happens, this does not enlarge the collection of functions we obtain through the apparently more restrictive unitary colligations, since any any contractive operator has a unitary dilation.

Lemma 3.2. *Let $W_\Sigma : X \rightarrow \mathcal{L}(\mathcal{H})$ be a transfer function obtained via a contractive colligation $\Sigma = (U, \mathcal{E}, \rho)$. Then there is unitary colligation $\tilde{\Sigma} = (\tilde{U}, \tilde{\mathcal{E}}, \tilde{\rho})$ such that $W_{\tilde{\Sigma}} = W_\Sigma$.*

Proof. At least one of the projections, say P_{λ_0} will be nonzero, so we take g to be a unit vector from its range. Let $\{a_j\}$ be an orthonormal basis for $\mathbb{C}^{n_{\lambda_0}}$, where $n_{\lambda_0} = 2^{|\lambda_0|-1}$, and define $\mathcal{Q} = \bigvee_j (g \otimes n_j)$. Elements of \mathcal{Q} have the form $e = \sum_j \beta_j g \otimes a_j$, where $\beta = (\beta_j) \in \mathbb{C}^{n_{\lambda_0}}$. In addition, if $e' = \sum_j \beta'_j g \otimes a_j$, then $\langle e', e \rangle = \sum_j \beta'_j \bar{\beta}_j$.

By assumption,

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{E} \oplus \mathcal{H} \rightarrow \mathcal{E} \oplus \mathcal{H}$$

is a contraction. Let

$$\tilde{D}_U = \begin{pmatrix} \tilde{D}_1 \\ \tilde{D}_2 \end{pmatrix} \quad \text{and} \quad D_U = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$$

be defect operators for U and U^* , respectively (so $1 - U^*U = \tilde{D}_U \tilde{D}_U^*$ and $1 - UU^* = D_U D_U^*$ with defect spaces $\tilde{\mathcal{D}}_U = \overline{\text{ran}} \tilde{D}_U^*$ and $\mathcal{D}_U = \overline{\text{ran}} D_U^*$), and $\begin{pmatrix} L^* & \tilde{D}_U^* \\ D_U & U \end{pmatrix}$ the corresponding Julia operator, which is unitary from $\tilde{\mathcal{D}}_U \oplus (\mathcal{E} \oplus \mathcal{H})$ to $\mathcal{D}_U \oplus (\mathcal{E} \oplus \mathcal{H})$. Then there is a unitary dilation of U of the form

$$U' = \overbrace{S_2 \oplus \cdots \oplus S_2}^{n_{\lambda_0}-1} \oplus \left(\begin{array}{cccccc|ccc} & & \ddots & & & & & \vdots & \\ & & & 1 & & & & & \\ & & & & 0 & & & & \\ & \ddots & & & & 1 & & & \\ & & 1 & & & & 0 & & \\ & & & 0 & & L^* & \tilde{D}_1^* & \tilde{D}_2^* & \\ & & & & 1 & 0 & 0 & 0 & \\ & & & & & 0 & D_1 & A & B \\ \hline \cdots & & & & & D_2 & C & D & \end{array} \right),$$

where unspecified entries are 0 and the blocks act on $(\mathcal{H}^{2n_{\lambda_0}-1} \oplus \mathcal{E}) \oplus \mathcal{H}$, $\mathcal{H} = \cdots \oplus \tilde{\mathcal{D}}_U \oplus \mathcal{D}_U \oplus \tilde{\mathcal{D}}_U \oplus \mathcal{D}_U$, a direct sum of defect spaces. The operator S_2 is a unitary operator on $\mathcal{H} \oplus \mathcal{H}$ defined as

$$S_2 = \left(\begin{array}{cccc|cccc} & & \ddots & & & & & \\ & & & 1 & & & & \\ & \ddots & & & 0 & & & \\ & & 1 & & & 1 & & \\ & & & 0 & & & 0 & \\ \hline & & & & 1 & & 0 & \\ & & & & & 1 & & \\ & & & & & & 0 & \\ & & & & & & & 1 \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & \ddots \end{array} \right).$$

(Here we have made the obvious identification of the direct sum defining \mathcal{H} written in the forward and backward direction with $\tilde{\mathcal{D}}_U$ and \mathcal{D}_U reversed.)

Let $\tilde{\mathcal{E}} = (\mathcal{H} \otimes \mathcal{E}) \oplus \mathcal{E}$. Define an isometry $Q : \mathcal{H} \otimes \mathcal{Q} \rightarrow \mathcal{H}^{2n_{\lambda_0}-1}$ by

$$Q(k \otimes e) = (\beta_1 k, 0, \beta_2 k, 0, \dots, \beta_{n_{\lambda_0}-1} k, 0, \beta_{n_{\lambda_0}} k), \quad \text{where } e = \sum_j \beta_j g \otimes a_j,$$

extending linearly. Let P to be the orthogonal projection onto $(\text{ran } Q^* \oplus \mathcal{E} \oplus \mathcal{H})^\perp$ in $\tilde{\mathcal{E}}$, and set

$$\tilde{U} = P \oplus (Q^* \oplus P_{\mathcal{E}} \oplus P_{\mathcal{H}}) U' (Q \oplus P_{\mathcal{E}} \oplus P_{\mathcal{H}}),$$

where $P_{\mathcal{E}}, P_{\mathcal{H}}$ are the orthogonal projections from $\tilde{\mathcal{E}} \oplus \mathcal{H}$ onto \mathcal{E} and \mathcal{H} . This is unitary on $\tilde{\mathcal{E}} \oplus \mathcal{H}$. We view it as a colligation by setting

$$\begin{aligned}\tilde{A} &= P \oplus (Q^* \oplus P_{\mathcal{E}})U'(Q \oplus P_{\mathcal{E}}) \\ \tilde{B} &= (Q^* \oplus P_{\mathcal{E}})U'P_{\mathcal{H}} \\ \tilde{C} &= P_{\mathcal{H}}U'(Q \oplus P_{\mathcal{E}}) \\ \tilde{D} &= P_{\mathcal{H}}U'P_{\mathcal{H}} = D.\end{aligned}$$

Define a unital representation

$$\tilde{\rho} = (1_{\mathcal{H}} \otimes \rho) \oplus \rho : C_b(\Lambda) \rightarrow \tilde{\mathcal{E}}.$$

Recall that using $\rho(f) = \sum_{\lambda} P_{\lambda} \otimes f$, we defined $S(x) = \sum_{\lambda} P_{\lambda} \otimes \sigma(x)$. If we now set $\tilde{P}_{\lambda} = (1_{\mathcal{H}} \otimes P_{\lambda}) \oplus P_{\lambda}$, we can likewise define

$$\tilde{S}(x) = \sum_{\lambda} \tilde{P}_{\lambda} \otimes \sigma(x) \in \mathcal{L}(\tilde{\mathcal{E}}),$$

and from this, a transfer function

$$\tilde{W}(x) = \tilde{D} + \tilde{C}\tilde{S}(x)(1_{\tilde{\mathcal{E}}} - \tilde{A}\tilde{S}(x))^{-1}\tilde{B}.$$

We verify that $\tilde{W}(x) = W(x)$ by showing $\tilde{C}\tilde{S}(x)(\tilde{A}\tilde{S}(x))^n \tilde{B} = CS(x)(AS(x))^n B$ for $n = 0, 1, \dots$. Fix $h \in \mathcal{H}$. Then

$$\tilde{B}h = (Q^* \oplus P_{\mathcal{E}}) \begin{pmatrix} \vdots \\ 0 \\ \tilde{D}_2^* h \\ 0 \\ Bh \end{pmatrix} = (k^0 \otimes e^0) \oplus Bh,$$

where $k^0 = (\dots \ 0 \ \tilde{D}_2^* h \ 0)^t$ and $e^0 = g \otimes a_1$ (since in the column vector, k^0 occurs in the first copy of \mathcal{H} in $\mathcal{H}^{2n_{\lambda_0}-1}$). Now

$$S(x)e^0 = S(x)(g \otimes a_1) = \sum_j \beta_j^1 h \otimes a_j$$

for some $(\beta_1^1, \dots, \beta_{n_{\lambda_0}}^1) \in \mathbb{C}^{n_{\lambda_0}}$. Setting $e^1 = \sum_j \beta_j^1 h \otimes a_j \in \mathcal{Q}$,

$$\tilde{S}(x)\tilde{B}h = (k^0 \otimes e^1) \oplus S(x)Bh.$$

Then

$$\tilde{C}\tilde{S}(x)\tilde{B}h = \begin{pmatrix} 0 & \dots & 0 & (\dots & 0 & D_2) & C \end{pmatrix} \begin{pmatrix} \beta_{n_{\lambda_0}}^1 k^0 \\ 0 \\ \vdots \\ \beta_1^1 k^0 \\ S(x)Bh \end{pmatrix} = CS(x)Bh,$$

proving the claim when $n = 0$.

For $n = 1$,

$$\begin{aligned}
& S_2 \oplus \cdots \oplus S_2 \oplus \begin{pmatrix} & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & \ddots & & & 1 & \\ & & 1 & & & 0 \\ & & & 0 & L^* & \tilde{D}_1^* \\ & & & 1 & 0 & 0 \\ & & & & 0 & D_1 & A \end{pmatrix} \begin{pmatrix} \beta_{n_{\lambda_0}}^1 k^0 \\ 0 \\ \vdots \\ \beta_1^1 k^0 \\ S(x)Bh \end{pmatrix} \\
&= \begin{pmatrix} \beta_{n_{\lambda_0}}^1 k^1 \\ 0 \\ \vdots \\ \beta_1^1 k^1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ k'^0 \\ AS(x)Bh \end{pmatrix},
\end{aligned}$$

where $k^1 = (\cdots \ 0 \ \tilde{D}_2^* h \ 0 \ 0 \ 0)^t$ (that is, k^0 with entries shifted up by two positions) and $k'^0 = (\cdots \ 0 \ \tilde{D}_1^* S(x)Bh \ 0)^t$. Notice that in both cases, only even numbered entries in odd numbered spaces \mathcal{H} of $\mathcal{H}^{2n_{\lambda_0}-1}$ are non-zero. Also, these vectors are in the kernel of P . From this, we conclude that

$$\tilde{A}\tilde{S}(x)\tilde{B}h = (k^1 \otimes e^1 + k'^0 \otimes e'^1) \oplus AS(x)Bh,$$

where e'^1 is likewise a vector in \mathcal{Q} . Applying $\tilde{S}(x)$, we get

$$(k^1 \otimes e^2 + k'^0 \otimes e'^2) \oplus S(x)AS(x)Bh$$

for some vectors e^2 and e'^2 in \mathcal{Q} . Because $(\cdots \ 0 \ D_2)$ only acts nontrivially on odd labeled entries of \mathcal{H} in the first \mathcal{H} of $\mathcal{H}^{2n_{\lambda_0}-1}$, we conclude that $\tilde{C}\tilde{S}(x)\tilde{A}\tilde{S}(x)\tilde{B}h = CS(x)AS(x)Bh$, proving the case when $n = 1$.

Repeated application of $\tilde{A}\tilde{S}(x)$ to vectors in $\mathcal{Q} \subset \mathcal{H}^{2n_{\lambda_0}-1}$ where the only nonzero entries are in the odd labeled spaces and within those spaces, only in the even labeled entries, yields vectors of the same sort. An induction argument then finishes the proof. \square

While we only stated and proved the last result in the specific case we need later in the paper, minor alterations would allow for it to cover cases where the test functions are operator valued (rather than simply matrix valued) and where there are infinitely many of them.

With Lemma 3.2 in hand, we can show that $\mathcal{T}(X, \Lambda, \mathcal{H})$ is a normed unital algebra.

Theorem 3.3. *With norm $\|\cdot\|$ defined as in (3.1), unit $1_X(x) = 1_{\mathcal{H}}$ and pointwise addition and multiplication, the set $\mathcal{T}(X, \Lambda, \mathcal{H})$ is a normed unital algebra. Furthermore, any $W \in \mathcal{T}(X, \Lambda, \mathcal{H})$ can be approximated uniformly in norm on compact subsets of X by elements of $\mathcal{L}(\mathcal{H}) \otimes \mathcal{P}_{\Psi}$, the operator valued polynomials in the test functions, while elements of $\mathcal{T}^A(X, \Lambda, \mathcal{H})$ can be approximated uniformly on X by such polynomials. If \mathcal{H} is finite dimensional, $\mathcal{L}(\mathcal{H}) \otimes \mathcal{P}_{\Psi}$ is dense in $\mathcal{T}(X, \Lambda, \mathcal{H})$ endowed with the supremum norm. Finally, if $W \in \mathcal{T}(X, \Lambda, \mathcal{H})$, $W = c(D + CS(1 - AS)^{-1}B)$, then W can be approximated uniformly in norm on compact subsets of X by polynomials in S which are in $c\mathcal{T}_1(X, \Lambda, \mathcal{H})$.*

Proof. We first show that $\mathcal{T}_1(X, \Lambda, \mathcal{H})$ is convex. Let $W_{\Sigma_1}, W_{\Sigma_2} \in \mathcal{T}_1(X, \Lambda, \mathcal{H})$ and $t \in [0, 1]$. The operator

$$\begin{aligned} U' &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t^{1/2} & (1-t)^{1/2} \end{pmatrix} \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{1/2} \\ 0 & 0 & (1-t)^{1/2} \end{pmatrix} \\ &= \begin{pmatrix} A_1 & 0 & t^{1/2} B_1 \\ 0 & A_2 & (1-t)^{1/2} B_2 \\ t^{1/2} C_1 & (1-t)^{1/2} C_2 & t D_1 + (1-t) D_2 \end{pmatrix}, \end{aligned}$$

being the product of contractions is a contraction. If we set $\mathcal{E}' = \mathcal{E}_1 \oplus \mathcal{E}_2$ and $\rho' = \rho_1 \oplus \rho_2$, then $t W_{\Sigma_1} + (1-t) W_{\Sigma_2} = W_{\Sigma'}$ where Σ' is a contractive colligation. Hence by the last lemma equals $t W_{\Sigma_1} + (1-t) W_{\Sigma_2} = W_{\Sigma}$ where Σ is some unitary colligation.

Clearly, by taking the contractive colligation with $U = 0$, the function which is identically 0 is in $\mathcal{T}_1(X, \Lambda, \mathcal{H})$. Hence by convexity, $t W_{\Sigma} \in \mathcal{T}_1(X, \Lambda, \mathcal{H})$ for all $t \in [0, 1]$, showing that $\mathcal{T}_1(X, \Lambda, \mathcal{H})$ is barreled.

Let $W_{\Sigma_1}, W_{\Sigma_2} \in \mathcal{T}_1(X, \Lambda, \mathcal{H})$ and define the unitary operator

$$U = \begin{pmatrix} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ C_1 & D_1 C_2 & D_1 D_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 & B_1 \\ 0 & 1 & 0 \\ C_1 & 0 & D_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_2 & B_2 \\ 0 & C_2 & D_2 \end{pmatrix}.$$

Let $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ and $\rho = \rho_1 \oplus \rho_2$, it follows that $W_{\Sigma_1} W_{\Sigma_2} = W_{\Sigma}$.

To see that what we defined in (3.1) is a norm, first of all note that if $W \in \mathcal{T}_1(X, \Lambda, \mathcal{H})$, then $\|W\| \leq 1$. It is also evident that $\|cW\| = |c|\|W\|$, and $\|W\| \geq 0$ with equality if and only if $W = 0$. Since $\mathcal{T}_1(X, \Lambda, \mathcal{H})$ is convex, if $W_1, W_2 \in \mathcal{T}_1(X, \Lambda, \mathcal{H})$ and $W_1 = c_1 W_{\Sigma_1}$, $W_2 = c_2 W_{\Sigma_2}$, then

$$\frac{c_1}{c_1 + c_2} W_{\Sigma_1} + \frac{c_2}{c_1 + c_2} W_{\Sigma_2} \in \mathcal{T}_1(X, \Lambda, \mathcal{H}).$$

Hence

$$\begin{aligned} \|W_1 + W_2\| &= \|c_1 W_{\Sigma_1} + c_2 W_{\Sigma_2}\| \\ &= (c_1 + c_2) \left\| \frac{c_1}{c_1 + c_2} W_{\Sigma_1} + \frac{c_2}{c_1 + c_2} W_{\Sigma_2} \right\| \\ &\leq c_1 + c_2. \end{aligned}$$

Taking the infimum over c_1 and c_2 as W_{Σ_1} and W_{Σ_2} range over those elements of $\mathcal{T}_1(X, \Lambda, \mathcal{H})$ such that $W_1 = c_1 W_{\Sigma_1}$ and $W_2 = c_2 W_{\Sigma_2}$ with $c_1, c_2 \geq 0$ yields the triangle inequality.

For any choice of representation $\rho : C_b(\Lambda) \rightarrow \mathcal{L}(\mathcal{E})$, and U the identity operator, we get $W_{\Sigma} = 1_X$, the function which is identically 1 $\mathcal{L}(\mathcal{H})$ on X . More generally, if $D \in \mathcal{L}(\mathcal{H})$ is a contraction operator,

and $U = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ for the same choice of \mathcal{E} and ρ , $W_{\Sigma} = D$.

If $\psi \in \Psi$ and we choose $\mathcal{E} = \mathcal{L}(\mathcal{H})$, $Z^+ = 1_{\mathcal{L}(\mathcal{H})}$ and $Z^- = \psi \otimes 1_{\mathcal{L}(\mathcal{H})}$, then $S = \psi \otimes 1_{\mathcal{L}(\mathcal{H})}$. So with $A = D = 0$ and $B = C = 1_{\mathcal{L}(\mathcal{H})}$, we get $W_{\Sigma} = \psi \otimes 1_{\mathcal{L}(\mathcal{H})}$. Then since $\mathcal{T}_1(X, \Lambda, \mathcal{H})$ is closed under products, we also have for any $n \in \mathbb{N}^{|\Psi|}$, $\psi^n \otimes 1_{\mathcal{L}(\mathcal{H})} \in \mathcal{T}_1(X, \Lambda, \mathcal{H})$. This also then gives that $\psi^n T$ for any contraction $T \in \mathcal{L}(\mathcal{H})$. Scaling and closure under addition yields that any operator valued polynomial in the test functions is in $\mathcal{T}(X, \Lambda, \mathcal{H})$.

The topology with which X is endowed ensures that all test functions are continuous. Hence for all $\lambda \in \Lambda$, ψ_{λ}^{\pm} is also continuous, and thus $\psi_{\lambda}^+ \psi_{\lambda}^{+*}$ is a continuous function bounded below by 1, and so has a continuous inverse. Consequently, any auxiliary test function σ_{λ} is continuous. (In

the case that Λ is an ample preordering, this was automatic, since $\sigma_\lambda \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathbb{C}^n})$ for some n , and all functions in this space are continuous.) Since for any $\lambda \in \Lambda$, $\|\sigma_\lambda(x)\| < 1$ for all $x \in X$, it follows that for any unitary colligation Σ , the associated function $S(x)$ is also continuous and has norm less than 1. Hence when $\mathcal{L}(\mathcal{H})$ is given the norm topology, $W_\Sigma \in \mathcal{T}_1(X, \Lambda, \mathcal{H})$ is continuous, and so $\mathcal{T}(X, \Lambda, \mathcal{H}) \subset C(X, \mathcal{L}(\mathcal{H}))$.

By definition, the test functions separate the points of X , and so by the Stone-Weierstrass theorem, the space of polynomials in the test functions, \mathcal{P}_Ψ , is dense in $\mathcal{T}(X, \Lambda, \mathbb{C})$ with the supremum norm. Hence if \mathcal{H} is finite dimensional with orthonormal basis (e_j) and $W \in \mathcal{T}(X, \Lambda, \mathcal{H})$, then $W_{j\ell} := \langle We_j, e_\ell \rangle \in \mathcal{T}(X, \Lambda, \mathbb{C})$. Let $\epsilon > 0$. For each $1 \leq j, \ell \leq \dim \mathcal{H}$, find a polynomial $p_{j\ell}$ such that $\|W_{j\ell} - p_{j\ell}\|_\infty < \epsilon/(\dim \mathcal{H})^2$. Then $\|W - (p_{j\ell})\|_\infty < \epsilon$, showing that $\mathcal{L}(\mathcal{H}) \otimes \mathcal{P}_\Psi$ is norm dense in $W \in \mathcal{T}(X, \Lambda, \mathcal{H})$. From this argument, we see that $\mathcal{L}(\mathcal{H}) \otimes \mathcal{P}_\Psi$ is weakly dense in $\mathcal{T}(X, \Lambda, \mathcal{H})$ if $\dim \mathcal{H}$ is not finite.

Now suppose that $W \in \mathcal{T}(X, \Lambda, \mathcal{H})$ where the dimension of \mathcal{H} is not necessarily finite. Fix $\epsilon > 0$ and let C be a compact subset of X . Then $W(C)$ is compact, and a cover of $W(C)$ by open balls in $\mathcal{L}(\mathcal{H})$ by balls of radius less than $\epsilon/12$ has a finite subcover $\{U_j\}$. For each j choose $x_j \in W^{-1}(U_j)$. Then for all $x \in X$, $\max_j \|W(x) - W(x_j)\| < \epsilon/6$.

For each j choose a finite dimensional subspace $\mathcal{H}_j \subset \mathcal{H}$ such that $\|W(x_j) - P_{\mathcal{H}_j} W(x_j)\|_{\mathcal{H}_j} < \epsilon/6$. Set $\mathcal{H}' = \bigvee_j \mathcal{H}_j$. This is finite dimensional and for all $x \in X$,

$$\begin{aligned} & \|W(x) - P_{\mathcal{H}'} W(x)\|_{\mathcal{H}'} \\ & \leq \max_j \|W(x) - W(x_j)\| + \max_j \|P_{\mathcal{H}'}(W(x) - W(x_j))\|_{\mathcal{H}'} + \max_j \|W(x_j) - P_{\mathcal{H}'} W(x_j)\|_{\mathcal{H}'} \\ & < \epsilon/2. \end{aligned}$$

As we have seen, we can find $p \in \mathcal{L}(\mathcal{H}') \otimes \mathcal{P}_\Psi$ such that $\|p - P_{\mathcal{H}'} W(x)\|_{\mathcal{H}'} < \epsilon/2$. Extending p to $\mathcal{L}(\mathcal{H}) \otimes \mathcal{P}_\Psi$ by padding with 0s, we then have that $\|W - p\| < \epsilon$, showing that we can approximate elements of $\mathcal{T}(X, \Lambda, \mathcal{H})$ pointwise, and hence uniformly in norm on compact subsets of X , by polynomials in $\mathcal{L}(\mathcal{H}) \otimes \mathcal{P}_\Psi$. If we know that $W \in \mathcal{T}^A(X, \Lambda, \mathcal{H})$, then by weak-* compactness of \bar{X} , we claim that we can approximate elements of $\mathcal{T}(X, \Lambda, \mathcal{H})$ uniformly in norm on X .

It suffices to prove the last claim in the case $c = 1$; that is, when $W = W_\Sigma$ for some colligation Σ . Let $\tilde{U} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$, where $\tilde{D} = D$, \tilde{A} is an $(M+2) \times (M+2)$ operator matrix with the first super-diagonal having all entries equal to A and all other entries 0, \tilde{B} is an $M+2$ operator column with the first $M+1$ entries equal to $\frac{1}{\sqrt{M+1}}B$ and the last entry 0, and \tilde{C} is an $M+2$ operator row with the first entry 0 and the remaining entries equal to $\frac{1}{\sqrt{M+1}}C$. It is easily verified that \tilde{U} is a contraction. Set \tilde{S} to the $(M+2) \times (M+2)$ diagonal matrix with diagonal entries equal to S . Then

$$W_M := \tilde{D} + \tilde{C}\tilde{S}(1 - \tilde{A}\tilde{S})^{-1}\tilde{B} \in \mathcal{T}_1(X, \Lambda, \mathcal{H})$$

and

$$\begin{aligned} W_M &= D + CS \left(\frac{M}{M+1} 1 + \frac{M-1}{M+1} AS + \cdots + \frac{1}{M+1} (AS)^M \right) B \\ &= D + CS(1 - AS)^{-1} \left(1 - \frac{1}{M+1} (1 - (AS)^{M+2})(1 - AS)^{-1} \right) B. \end{aligned} \tag{3.2}$$

Since by Lemma 2.21 $S(x)$ is a strict contraction, we see that W_M converges pointwise with M to W . Arguing as above, we then get W_M converging uniformly on compact subsets of X to W . \square

We write $\overline{\mathcal{T}}(X, \Lambda, \mathcal{H})$ for the completion of $\mathcal{T}(X, \Lambda, \mathcal{H})$ in the norm from (3.1), and $\overline{\mathcal{T}}^A(X, \Lambda, \mathcal{H})$ for the closure of $\mathcal{T}^A(X, \Lambda, \mathcal{H})$ in $\overline{\mathcal{T}}(X, \Lambda, \mathcal{H})$.

Corollary 3.4. *The spaces $\{\overline{\mathcal{T}}(X, \Lambda, \mathcal{H} \otimes M_n(\mathbb{C}))\}_{n \in \mathbb{N}}$ and $\{\overline{\mathcal{T}}^A(X, \Lambda, \mathcal{H} \otimes M_n(\mathbb{C}))\}_{n \in \mathbb{N}}$ define unital operator algebra structures for $\overline{\mathcal{T}}(X, \Lambda, \mathcal{H})$ and $\overline{\mathcal{T}}^A(X, \Lambda, \mathcal{H})$, respectively.*

Proof. Let $W \in \mathcal{T}_1(X, \Lambda, \mathbb{C}^n \otimes \mathcal{H})$ with unitary colligation $\Sigma = (U, \mathcal{H}, \rho)$, $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and let $X \in M_{mn}(\mathbb{C})$, $Y \in M_{nm}(\mathbb{C})$ with $\|X\|, \|Y\| \leq 1$. Then

$$XWY = XDY + XC(1 - AS)^{-1}BY = \tilde{W}$$

where $\tilde{W} \in \mathcal{T}_1(X, \Lambda, \mathcal{H} \otimes \mathbb{C}^m)$ has contractive colligation $\tilde{\Sigma} = (\tilde{U}, \mathcal{H}, \rho)$ with $\tilde{U} = \begin{pmatrix} A & BY \\ XC & XDY \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix}$. Hence $\overline{\mathcal{T}}(X, \Lambda, \mathcal{H})$ is an abstract operator space. Since for all n , $W_1, W_2 \in \mathcal{T}_1(X, \Lambda, \mathcal{H} \otimes \mathbb{C}^n)$ implies $W_1 W_2 \in \mathcal{T}_1(X, \Lambda, \mathcal{H} \otimes \mathbb{C}^n)$, it follows that $\overline{\mathcal{T}}(X, \Lambda, \mathcal{H})$ is an operator algebra. The case for $\overline{\mathcal{T}}^A(X, \Lambda, \mathcal{H})$ is proved similarly. \square

The above provides something of a converse to the main result of Jury [32] in a special case.

We close this subsection with a lemma which will be useful when we want to construct representations on algebras of transfer functions.

Lemma 3.5. *Let $W_\Sigma : X \rightarrow \mathcal{L}(\mathcal{H})$ be a transfer function obtained via a unitary colligation $\Sigma = (U, \mathcal{E}, \rho)$. Then there is another unitary colligation $\tilde{\Sigma} = (\tilde{U}, \tilde{\mathcal{E}} = \mathcal{E} \otimes \mathcal{H}, \tilde{\rho})$ such that $W_{\tilde{\Sigma}} = W_\Sigma$.*

Proof. Recall that by construction, there are orthogonal projections P_λ with orthogonal ranges such that $\mathcal{E} = \bigoplus_\lambda \text{ran } P_\lambda \otimes \mathbb{C}^{n_\lambda}$, $n_\lambda = 2^{|\lambda|-1}$, and $S(x) = \sum_\lambda P_\lambda \otimes \sigma_\lambda(x)$. We construct the new colligation from the old by taking

$$\tilde{\mathcal{E}} = \bigoplus_\lambda \text{ran } P_\lambda \otimes (\mathbb{C}^{n_\lambda} \otimes \mathcal{H}),$$

and setting

$$\tilde{S}(x) = \sum_\lambda P_\lambda \otimes (\sigma_\lambda(x) \otimes 1_{\mathcal{L}(\mathcal{H})}).$$

Fix $e \in \mathcal{H}$ with $\|e\| = 1$. Define an operator $\tilde{U} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$ on $\tilde{\mathcal{E}} \oplus \mathcal{H}$ as follows. For $f \in \mathcal{E}$, $h, g \in \mathcal{H}$ decomposed as $g = \alpha e + e^\perp$ where $\langle e, e^\perp \rangle = 0$. Then set

$$\tilde{A}(f \otimes \alpha e + f \otimes e^\perp) := Af \otimes \alpha e + f \otimes e^\perp,$$

$$\tilde{B}h := Bh \otimes e,$$

$$\tilde{C}(f \otimes \alpha e + f \otimes e^\perp) := \alpha Cf$$

$$\tilde{D}h := Dh,$$

extending by linearity where necessary. One easily checks that the adjoints of these operators are given by

$$\tilde{A}^*(f \otimes \alpha e + f \otimes e^\perp) = Af \otimes \alpha e + f \otimes e^\perp,$$

$$\tilde{B}^*(f \otimes \alpha e + f \otimes e^\perp) = \alpha B^*f,$$

$$\tilde{C}^*h = C^*h \otimes e$$

$$\tilde{D}^*h = D^*h,$$

again extending by linearity as needed. A straightforward calculation gives

$$\begin{aligned} (\tilde{A}^* \tilde{A} + \tilde{C}^* \tilde{C})(f \otimes \alpha e + f \otimes e^\perp) &= (A^* A + C^* C)f \otimes \alpha e + f \otimes e^\perp = f \otimes \alpha e + f \otimes e^\perp, \\ (\tilde{B}^* \tilde{B} + \tilde{D}^* \tilde{D})h &= (B^* B + D^* D)h, \end{aligned}$$

showing that the operators so defined are bounded. The other equations needed to show that \tilde{U} is unitary are likewise checked.

We find that $\tilde{C}\tilde{S}(x)\tilde{B}h = \tilde{C}\tilde{S}(x)(Bh \otimes e) = \tilde{C}((S(x)Bh) \otimes e) = CS(x)Bh$. Also, $\tilde{C}(\tilde{A}\tilde{S}(x))^n \tilde{B}h = C(AS(x))^n Bh$. We conclude that $W_\Sigma = W_\Sigma$. \square

3.2. Contractivity and complete contractivity of representations of transfer function algebras.

Definition 3.6. We write that a representation $\pi : \mathcal{T}^A(X, \Lambda, \mathcal{H}) \rightarrow \mathcal{L}(\mathcal{G})$ or $\pi : \mathcal{T}(X, \Lambda, \mathcal{H}) \rightarrow \mathcal{L}(\mathcal{G})$ is **contractive on auxiliary test functions** if for each $\lambda \in \Lambda$, an appropriate ampliation of π (also denoted by π) has the property that $\pi(\sigma_\lambda \otimes 1_{\mathcal{L}(\mathcal{H})}) \leq 1$. It is said to be strictly contractive in case this is a strict inequality. A representation is **strongly / weakly continuous** if whenever a bounded net (φ_α) converges pointwise in norm to φ (in other words, $\sup_\alpha \|\varphi_\alpha\| < \infty$ and for each $x \in X$, $\|\varphi_\alpha(x) - \varphi(x)\| \rightarrow 0$), then $\pi(\varphi_\alpha)$ converges strongly / weakly to $\pi(\varphi)$.

Given a bounded unital representation π of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, we define $\pi(\psi_\lambda^\pm)$ by applying π entry-wise. Then π is a Brehmer representation if and only if π is contractive on the test functions and for any maximal element λ of the preordering Λ ,

$$\pi(\psi_\lambda^+) \pi(\psi_\lambda^+)^* - \pi(\psi_\lambda^-) \pi(\psi_\lambda^-)^* \geq 0.$$

In this case, for each λ in the maximal preordering associated to Λ , there is a contraction $G : \text{ran } \pi(\psi_\lambda^+)^* \rightarrow \pi(\psi_\lambda^-)^*$ such that $\pi(\psi_\lambda^+)G_\lambda = \pi(\psi_\lambda^-)$. The following is then well defined:

$$\pi(\sigma_\lambda) = G_\lambda,$$

though properly speaking, this should be viewed as an ampliation of the representation π . As we saw in Theorem 2.19, when Λ is an ample preordering, we can extend σ_λ to a function in $H^\infty(X, \mathcal{K}_{\Lambda, \mathbb{C}^n})$ where $n = 2^{|\lambda|-1}$, and so π (or rather $\pi^{(n)}$) is already defined on σ_λ , and potentially may not be equal to G_λ . Nevertheless, it is the case that once π is given on test functions, it induces a well defined map which is contractive on auxiliary test functions, and so on the algebra of transfer functions, as we shall see.

The next theorem is a version of the von Neumann inequality for the algebra $\mathcal{T}(X, \Lambda, \mathcal{H})$.

Theorem 3.7. *Let $\pi : \mathcal{T}^A(X, \Lambda, \mathcal{H}) \rightarrow \mathcal{L}(\mathcal{G})$ be a unital representation which is contractive on auxiliary test functions, or $\pi : \mathcal{T}(X, \Lambda, \mathcal{H}) \rightarrow \mathcal{L}(\mathcal{G})$ be a weakly continuous unital representation which is contractive on auxiliary test functions. For all W_Σ in $\mathcal{T}_1^A(X, \Lambda, \mathcal{H})$ (respectively, $\mathcal{T}_1(X, \Lambda, \mathcal{H})$), $\|\pi(W_\Sigma)\| \leq 1$; that is π is contractive.*

Proof. We begin by observing that in either case, the representation $\pi_0 : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{G})$ obtained by restricting π to constant functions is a unital representation of the C^* -algebra $\mathcal{L}(\mathcal{H})$, and so is contractive. The same is true of the ampliations of π_0 , so it is in fact completely contractive.

Let $W = W_\Sigma \in \mathcal{T}_1(X, \Lambda, \mathcal{H})$, where $\Sigma = (U, \mathcal{E}, \rho)$, $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, is a unitary colligation. For $r \in (0, 1)$, define $W_r = W_{\Sigma_r}$, where $\Sigma_r = (U_r, \mathcal{H}, \rho)$ is a contractive colligation with

$$U_r = U \begin{pmatrix} r1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} rA & B \\ rC & D \end{pmatrix}.$$

By Lemma 3.2, $W_r \in \mathcal{T}_1(X, \Lambda, \mathcal{H})$, and

$$W_r = D + C(rS)(1 - A(rS))^{-1}B$$

We now follow the line of proof in Lemma 3.1 of [28]. Since $rAS(x)$ is a strict contraction, $\frac{M}{M+1} \sum_{n=1}^M \frac{M-n}{M} (rAS(x))^n$ converges uniformly in norm to $(1 - rAS(x))^{-1}$, and by the proof of the last statement in Theorem 3.3, for all M , $W_{r,M} := D + CS \left(\frac{M}{M+1} 1 + \frac{M-1}{M+1} AS + \dots + \frac{1}{M+1} (AS)^M \right) B \in \mathcal{T}_1(X, \Lambda, \mathcal{H})$ and converges pointwise in norm with M to W_r . By assumption then, $\pi(W_{r,M})$ converges weakly to $\pi(W_r)$.

As in the proof of Lemma 3.1 of [28], we see that $\pi(W_r) = W_{\tilde{\Sigma}_r} \in \mathcal{T}_1(X, \Lambda, \mathcal{G})$ with $\tilde{\Sigma}_r = (\tilde{U}_r, \mathcal{E} \otimes \mathcal{G}, \rho \otimes \pi)$, where $\tilde{U}_r = \begin{pmatrix} r\tilde{A} \otimes 1 & \tilde{B} \otimes 1 \\ r\tilde{C} \otimes 1 & \tilde{D} \otimes 1 \end{pmatrix}$. We have $\tilde{D} = \pi(D)$ and \tilde{A} , \tilde{B} and \tilde{C} are obtained by applying π component-wise. Since $\pi_0 = \pi|_{\mathcal{L}(\mathcal{H})}$ is completely contractive, \tilde{U}_r is a contraction. Hence $\|\pi(\varphi_r)\| \leq 1$. Now $(\varphi_r)_r$ is a bounded net converging pointwise in norm to φ , so by assumption $(\pi(\varphi_r))_r$ converges weakly to $\pi(\varphi)$, meaning that $\|\pi(\varphi)\| \leq 1$.

For $\mathcal{T}^A(X, \Lambda, \mathcal{H})$, the same argument applies when π is simply assumed to be contractive on auxiliary test functions, since $\mathcal{T}^A(X, \Lambda, \mathcal{H})$ is the norm closure of polynomials in test functions. \square

Corollary 3.8. *Let π be a representation of $\mathcal{T}^A(X, \Lambda, \mathcal{H})$, respectively, a weakly continuous representation of $\mathcal{T}(X, \Lambda, \mathcal{H})$, which is contractive on auxiliary test functions. Then π is completely contractive.*

Proof. If π is a representation of either $\mathcal{T}^A(X, \Lambda, \mathcal{H})$ or $\mathcal{T}(X, \Lambda, \mathcal{H})$ which is contractive on auxiliary test functions, then the same is true for $\pi^{(n)}$ for all n . Hence the result follows from the previous theorem applied to the auxiliary test functions tensored with 1_n . \square

3.3. Brehmer representations and spectral sets.

Definition 3.9. Let π be a bounded unital representation of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. Call π a **Brehmer representation** (associated to the preordering Λ) if for any test function ψ , $\|\pi(\psi \otimes 1_{\mathcal{L}(\mathcal{H})})\| \leq 1$ and for all $\lambda \in \Lambda$,

$$\prod_{\lambda \ni \lambda_i \neq 0} (1 - \pi(\psi_i \otimes 1_{\mathcal{L}(\mathcal{H})}) \pi(\psi_i \otimes 1_{\mathcal{L}(\mathcal{H})})^*)^{\lambda_i} \geq 0. \quad (3.3)$$

Note since $\mathcal{L}(\mathcal{H})$ is a C^* -algebra, it is automatic that $\pi_0 = \pi|_{\mathcal{L}(\mathcal{H})}$ with $\pi_0(T) = \pi(1 \otimes T)$ is completely contractive.

A representation π of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ is a **strict Brehmer representation** if the inequalities in (3.3) are strict. It is a **strongly / weakly continuous Brehmer representation** if it is a Brehmer representation and which is either strongly or weakly continuous in the sense defined in the last subsection.

We say that X is a **spectral set** for the representation π (equivalently, that the **von Neumann inequality** holds) if π is a contractive representation of $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. It is a **complete spectral set** if π is a completely contractive representation of $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$.

A representation $\tilde{\pi}$ **dilates** a representation π (equivalently, π **dilates to** $\tilde{\pi}$) if π is the restriction of $\tilde{\pi}$ to a semi-invariant subspace; that is, the difference of two invariant subspaces. The **H^∞ dilation property** is said to hold for a domain X if whenever π is a representation of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ for which X is a spectral set, then X is a complete spectral set for π .

While Brehmer representations induce representations which are contractive on test functions, the converse is also true.

Lemma 3.10. *If a representation π of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ is contractive on auxiliary test functions then it is a Brehmer representation.*

Proof. This follows from (2.4). □

Clearly, a strict Brehmer representation is norm continuous, a norm continuous one is strongly continuous, and a strongly continuous one is weakly continuous. The H^∞ dilation property is akin to the better known **rational dilation property**, where $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ is replaced by the algebra of functions generated by the rational functions over a compact subset of \mathbb{C}^d with poles off of the set.

The connection of the von Neumann inequality as defined above with the usual von Neumann inequality is as follows. Suppose that $X = \mathbb{D}^d$ and Ψ is the set of coordinate functions in \mathbb{C}^d (so $\psi_j(z) = z_j$ for $j = 1, \dots, d$), and assume that $\Lambda = \{e_j\}_{j=1, \dots, d}$. Then Agler's realization theorem for the polydisk (Theorem 2.7 above) implies that any representation π of $H^\infty(X, \mathcal{K}_\Lambda)$ for which $T_j = \pi(\psi_j)$ is strictly contractive for all j (so (T_1, \dots, T_d) is a tuple of commuting strict contractions) is contractive on $H^\infty(X, \mathcal{K}_\Lambda)$. Note that in this case $S(z) = Z^-(z) = \sum_j P_j z_j$, where P_j s are orthogonal projections summing to the identity. We therefore naturally take $\pi(S(z)) = \sum_j P_j \otimes T_j$, which then, via the transfer function representation, allows us to interpret $\pi(\varphi)$ for $\varphi \in H_1^\infty(X, \mathcal{K}_\Lambda)$ in the natural way. So in other words, for a tuple T of commuting operators with $\|T_j\| < 1$ for all j , $\|\varphi(T)\| \leq 1$ for all φ in the Schur-Agler class of the polydisk.

The name for the rational dilation property derives from a theorem of Arveson [10], which states in the example from the previous paragraph, a tuple T of commuting contractions has a commuting unitary dilation U if and only if for all $n \in \mathbb{N}$, T induces a completely contractive representation π on the algebra \mathcal{P} of polynomials over \mathbb{C}^d , the norm closure of which is the polydisk analogue of the disk algebra. Write $\tilde{\pi}$ for the representation induced by U . By the spectral theorem for normal operators, $\tilde{\pi}$ is completely contractive. The converse direction is an application of the Arveson extension theorem and Stinespring dilation theorem. Of course there would be no hope of dilating T to U if it were the case that the representation induced by T is not contractive, which the example due to Kaijser and Varopoulos [46] demonstrates can happen when $d \geq 3$ in $H^\infty(\mathbb{D}^d)$.

Because \mathbb{D}^d is polynomially convex, the polynomial algebra suffices when considering rational dilation in this setting. For more complex domains $X \subset \mathbb{C}^d$ such as for example an annulus in \mathbb{C} , one needs to consider $M_n(\mathbb{C})$ valued rational functions over \mathbb{C}^d with poles off of \bar{X} , and the commuting tuple of unitary operators is replaced by a commuting tuple of normal operators with spectrum supported on ∂X (or more precisely, the distinguished boundary of X).

It becomes evident then that one can view Arveson's theorem as describing when a contractive representation of the analogue of the disk algebra is completely contractive. An example due to Parrott [40] shows that when $d \geq 3$, there are contractive representations which are not completely contractive. Further examples when $d = 3$ are given by Bagchi, Bhattacharyya and Misra in [12], and they show that these examples are not even 2-contractive. As we shall see, this is no accident — in fact any representation which is contractive but not completely contractive must fail to be 2-contractive.

When $d = 1$ or 2, contractive representations are automatically completely contractive by the Sz.-Nagy dilation theorem and Andô's theorem, respectively. Agler showed that over an annulus \mathbb{A} , it is again the case that contractive representations of the algebra of functions analytic in a neighborhood of \mathbb{A} are completely contractive. This was later shown to fail for domains of higher connectivity [3, 27, 41].

It is a consequence of the Arveson extension theorem and the Stinespring dilation theorem that any completely contractive representation of either $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ or $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ extends to a completely contractive representation of $C^*(H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}}))$ or $C^*(A(X, \mathcal{K}_{\Lambda, \mathcal{H}}))$, respectively.

We have the following dilation theorem, generalizing Arveson's dilation result for the polydisk.

Theorem 3.11. *Let π be a representation of $\mathcal{T}^A(X, \Lambda, \mathcal{H})$, or a weakly continuous representation of $\mathcal{T}(X, \Lambda, \mathcal{H})$, which is contractive on auxiliary test functions. Then π dilates to a completely contractive representation $\tilde{\pi}$ of $C^*(\mathcal{T}(X, \Lambda, \mathcal{H}))$ (respectively, $C^*(\mathcal{T}^A(X, \Lambda, \mathcal{H}))$), with the property that the only completely positive map agreeing with $\tilde{\pi}$ on $\mathcal{T}(X, \Lambda, \mathcal{H})$ (respectively, $\mathcal{T}^A(X, \Lambda, \mathcal{H})$) is $\tilde{\pi}$ itself.*

Proof. This is a corollary of Corollary 3.8 and Theorem 1.1 of [29]. \square

A representation with the properties of $\tilde{\pi}$ (ie, that $\tilde{\pi}$ extends uniquely as a completely positive map to the C^* -envelope) is called a **boundary representation** if, in addition, it is irreducible. We use an alternative, equivalent description of boundary representations due Muhly and Solel [39] below.

An analogue of the rational dilation problem ask whether every contractive representation of $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ is completely contractive. Likewise, one might ask if every contractive representation of $H^\infty(X, \mathcal{K}_\Lambda)$ (or more generally, of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$) is automatically completely contractive; that is, whether the H^∞ dilation property holds. Perhaps surprisingly, even for $H^\infty(\mathbb{D})$ this is unknown. The problem is that in many cases the boundary of X is rather complicated, since it is the difference between the Stone-Ćech compactification of X and X in the appropriate topology, and this can be very complex. There will be representations corresponding to point evaluations in the boundary. In general, these may not be weak-* continuous, and so there is no obvious characterization of contractive representations of $H^\infty(X, \mathcal{K}_\Lambda)$ in terms of its action on test functions, which is generally what is used in the showing the contractivity of ampliations of a representation.

As an alternative, one might ask if there are any simply described subclasses of the contractive representations which are completely contractive. For example, we will prove that representations of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ which are Brehmer representations and which are weakly continuous are completely contractive. We should note that for general Λ , it is easy to find examples where not all contractive representations are Brehmer representations.

Over \mathbb{D}^d when $d \geq 3$, Parrott's example implies that rational dilation fails for $A(\mathbb{D}^3)$, though as we saw in Corollary 3.4, with the Agler algebra and Schur-Agler matrix norm structure, this is not the case. We prove that in general any representation of $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ which is contractive on the auxiliary test functions is completely contractive. When the preordering is ample over d test functions, this will imply that any representation which is 2^{d-1} -contractive is completely contractive. As we will show, there is an improvement which can be made to this when $d > 1$ using the so-called nearly ample preorderings, and giving that 2^{d-2} -contractive representations are completely contractive. In particular, this will imply that for $d \geq 3$, 2^{d-2} -contractive representations of $A(\mathbb{D}^d)$ are completely contractive, and that such representations of $H^\infty(\mathbb{D}^d)$ which are at least weakly continuous are also completely contractive. When $d = 3$ then, 2 contractivity will imply complete contractivity, and so any example like Parrott's of a contractive representation of $A(\mathbb{D}^3)$ which is contractive but not completely contractive must fail to be 2-contractive.

3.4. Some boundary representations for the classical Agler algebra. Since in the classical setting the auxiliary test functions are simply the test functions, it follows from Theorem 2.13 that any representation of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ which is contractive is completely contractive. At first this may seem to contradict the examples of Parrott [40] and Varopoulos and Kaiser [46] when $X = \mathbb{D}^3$, which both give commuting tuples of contractions on $H^\infty(\mathbb{D}^3)$ which do not dilate to commuting unitary operators (indeed, the Kaijser-Varopoulos example is not even a contractive representation of $H^\infty(\mathbb{D}^3)$). The reason that there is no difficulty is that the Schur-Agler norm of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ (and more generally, the corresponding matrix norm structure) is not the same as the supremum norm in this case.

Let us consider more closely the classical Agler algebra over the tridisk. We examine the representations generated by commuting triples of contractions from several particularly interesting examples: first that of Parrott, then a Kaijser-Varopoulos type example due to Grinshpan, Kaliuzhnyi-Verbovetskyi and Woerdeman from [31], and finally the Kaijser-Varopoulos example itself. We show that these give rise to nontrivial non-scalar boundary representations for the disk algebra analogue for the classical Agler algebra. Of course such representations are expected since, as has been noted [21], this is not a uniform algebra, but these are explicit. According to a result of Muhly and Solel [39], a **boundary representation** in the sense of Arveson is an irreducible completely contractive unital representation of $H^\infty(X, \mathcal{K}_\Lambda)$ with the property that any completely contractive dilation of this representation must contain it as a direct summand (see also [29]).

We begin by considering the Parrott example.

Lemma 3.12. *Let $X = \mathbb{D}^3$, $\Psi = \{z_1, z_2, z_3\}$ a collection of test functions on X , $\Lambda = \{e_1, e_2, e_3\}$, and \mathcal{K} the corresponding set of admissible kernels. Let $U, V \in \mathcal{L}(\mathcal{K})$ be unitary operators with the property that $UV = -VU$ (for example, we might choose $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). Then on $\mathcal{K} \oplus \mathcal{K}$,*

$$\pi(z_1) := T_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi(z_2) := T_2 = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \quad \pi(z_3) := T_3 = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix},$$

defines a (completely contractive) boundary representation of $H^\infty(X, \mathcal{K}_\Lambda)$.

Proof. It is obvious that the operators in the statement of the lemma commute. By Theorem 2.13, this gives a contractive representation of $H^\infty(X, \mathcal{K}_\Lambda)$, and so by Corollary 3.4 a completely contractive representation. It is clearly irreducible. As noted in the discussion preceding the statement of the lemma, it suffices to prove that any contractive dilation of this representation contains it as a direct summand.

Assume that

$$\tilde{\pi}(z_1) = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & T_1 & A_4 \\ 0 & 0 & A_5 \end{pmatrix}, \quad \tilde{\pi}(z_2) = \begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & T_2 & B_4 \\ 0 & 0 & B_5 \end{pmatrix}, \quad \tilde{\pi}(z_3) = \begin{pmatrix} C_1 & C_2 & C_3 \\ 0 & T_3 & C_4 \\ 0 & 0 & C_5 \end{pmatrix}$$

are commuting contractions. We show that A_2, B_2, C_2, A_4, B_4 and C_4 are zero. Since $1, U$ and V are unitary, it follows that

$$A_2 = \begin{pmatrix} a & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} b & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} c & 0 \end{pmatrix}$$

on $\mathcal{K} \oplus \mathcal{K}$. Commutativity then gives

$$A_1 B_2 + A_2 T_2 = B_1 A_2 + B_2 T_1$$

$$A_1 C_2 + A_2 T_3 = C_1 A_2 + C_2 T_1$$

$$B_1 C_2 + B_2 T_3 = C_1 B_2 + C_2 T_2.$$

Right multiplication of the first of these by T_1 yields $A_1 b = B_1 a$, and so $A_1 B_2 = B_1 A_2$. Hence $A_2 T_2 = B_2 T_1$, and so $aU = b$. Similar calculations with the other two equations give $aV = c$ and $bV = cU$. Thus $aUV = bV = cU = aVU = -aUV$, and since UV is unitary, $a = 0$. We then also have $b = c = 0$. A similar calculation shows that A_4, B_4 and C_4 are zero. \square

We next turn to the example of Grinshpan, Kaliuzhnyi-Verbovetskyi and Woerdeman from [31], which again as in the Parrott example is nilpotent, but this time of order 2.

Theorem 3.13. Let $X = \mathbb{D}^3$, $\Psi = \{z_1, z_2, z_3\}$ a collection of test functions on X , $\Lambda = \{e_1, e_2, e_3\}$, and \mathcal{K} the corresponding set of admissible kernels. Let $u_1, u_2, u_3 \in \mathbb{R}^2$ be unit vectors with the property that $u_1 + u_2 + u_3 = 0$ (without loss of generality, we may assume $u_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} \sqrt{3}/2 & -1/2 \end{pmatrix}$, $u_3 = \begin{pmatrix} -\sqrt{3}/2 & -1/2 \end{pmatrix}$). Define a representation $\pi : H^\infty(X, \mathcal{K}_\Lambda) \rightarrow M_4(\mathbb{C})$ by

$$\pi(z_j) := T_j = \begin{pmatrix} 0 & u_j & 0 \\ 0 & 0 & u_j^* \\ 0 & 0 & 0 \end{pmatrix}, \quad j = 1, 2, 3.$$

Then this is a (completely contractive) boundary representation of $H^\infty(X, \mathcal{K}_\Lambda)$.

Proof. We assume that we have made the explicit choice of u_j s mentioned in the statement of the theorem. Consider a commuting contractive dilation

$$V_j = \begin{pmatrix} a_j & b_j & v_j & c_j & d_j \\ 0 & 0 & u_j & 0 & e_j \\ 0 & 0 & 0 & u_j^* & v_j'^* \\ 0 & 0 & 0 & 0 & f_j \\ 0 & 0 & 0 & 0 & g_j \end{pmatrix}, \quad j = 1, 2, 3,$$

of the T_j s. Because each u_j is a unit vector, $c_j = 0$ and $e_j = 0$ for each j . We also have that $u_j v_j'^* = u_j v_j'^* = 0$, so

$$\begin{aligned} v_1 &= \alpha_1 \begin{pmatrix} 1 & 0 \end{pmatrix} & v_2 &= \alpha_2 \begin{pmatrix} -1/2 & -\sqrt{3}/2 \end{pmatrix} & v_3 &= \alpha_3 \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix} \\ v_1' &= \alpha_1' \begin{pmatrix} 1 & 0 \end{pmatrix} & v_2' &= \alpha_2' \begin{pmatrix} -1/2 & -\sqrt{3}/2 \end{pmatrix} & v_3' &= \alpha_3' \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}. \end{aligned}$$

By commutativity,

$$u_j v_k'^* = u_k v_j'^* \quad \text{and} \quad v_j u_k^* = v_k u_j^*.$$

Using the explicit form of these vectors, it is easy to check that the first of these equations gives $\alpha_2' = \alpha_3' = -\alpha_1'$ and $\alpha_2' = -\alpha_3'$, and so $\alpha_j' = 0$ for all j . Similar calculations with the second equation yields $\alpha_j = 0$ for all j as well. Thus $v_j = v_j' = 0$ for all j .

It also follows from commutativity that $b_j u_k = b_k u_j$, and since the u_k s are pairwise linearly independent, it follows that $b_j = 0$ for all j . Likewise, $f_j = 0$ for all j , and so we conclude that each V_j contains T_j as a direct summand.

Finally, we show that the representation is irreducible. If $\mathcal{G} \subset \mathbb{R}^4$ is a reducing subspace, then it is invariant for $T_j^* T_j$ and $T_j T_j^*$ for each j . From this we see that $\mathcal{G} \neq \mathbb{C}^4$, any vector in \mathcal{G} must be of the form $v_1 = \begin{pmatrix} c_1 & 0 & 0 & c_2 \end{pmatrix}^t$, $v_2 = \begin{pmatrix} c_1 & c_2 & c_3 & 0 \end{pmatrix}^t$, $v_3 = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \end{pmatrix}^t$, where $c_j \in \mathbb{C}$ for all j . Multiplying v_1 by T_j we get $c_2 = 0$, and by T_j^* we get $c_1 = 0$; that is, $\mathcal{G} = \{0\}$. Similarly, since the u_j s span \mathbb{R}^2 , we conclude after considering $T_j^* v_2$ and $T_j v_3$ that $c_2 = c_3 = 0$ in the first case and $c_1 = c_2 = 0$ in the second, and from this that $c_1 = 0$ in v_2 and $c_3 = 0$ in v_3 , finishing the proof. \square

Finally, we turn to the Kaijser-Varopoulos example. As it happens, the operators there can be dilated to other commuting contractions which can only be further dilated by means of a direct sum. The proof is similar to the above, and we leave it as an exercise for the interested reader.

Theorem 3.14. Let $X = \mathbb{D}^3$, $\Psi = \{z_1, z_2, z_3\}$ a collection of test functions on X , $\Lambda = \{e_1, e_2, e_3\}$, and \mathcal{K} the corresponding set of admissible kernels. Then the representation $\pi : H^\infty(X, \mathcal{K}_\Lambda) \rightarrow M_6(\mathbb{C})$ defined

by

$$\begin{aligned}\pi(z_1) := T_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 \end{pmatrix}, \\ \pi(z_2) := T_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 \end{pmatrix}, \\ \pi(z_3) := T_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & 0 & 0 \end{pmatrix}.\end{aligned}$$

is a (completely contractive) boundary representation of $H^\infty(X, \mathcal{K}_\Lambda)$.

We finally mention that the Crabb-Davie example [23] gives a boundary representation of the Agler algebra consisting of nilpotent operators of order 3, and presumably examples with any degree of nilpotency can be constructed in a similar manner. Using results of [36] and [5], one can show that there will be boundary representations of the Agler algebra which are neither commuting unitaries nor commuting nilpotents.

4. REALIZATION THEOREMS

4.1. The first realization theorem. As usual, we assume all test functions are in $A(X, \mathcal{K}_\Lambda)$.

Fix a finite set $F \subset X$. Define a cone in $M_{|F|}(\mathbb{C})$ by

$$\mathcal{C}_F := \left\{ \left(\Gamma(x, y) \left(E^+(x)E^+(y)^* - E^-(x)E^-(y)^* \right) \right) : \Gamma \in \mathbb{K}_X^+(C_b(\Lambda), \mathbb{C}) \right\}.$$

This is a cone rather than simply a wedge since $E^+(x)E^+(x)^* - E^-(x)E^-(x)^* > 0$, and so if $\Gamma_1, \Gamma_2 \geq 0$ with $(\Gamma_1(x, y) \left(E^+(x)E^+(y)^* - E^-(x)E^-(y)^* \right)) = -(\Gamma_2(x, y) \left(E^+(x)E^+(y)^* - E^-(x)E^-(y)^* \right))$ for all $x, y \in F$, then for all x , $\Gamma_1(x, x) = \Gamma_2(x, x) = 0$, and hence by positivity, $\Gamma_1(x, y) = \Gamma_2(x, y) = 0$ for all $x, y \in F$.

More generally, there is an operator version of this. For a fixed Hilbert space \mathcal{H} , define a cone in $M_{|F|}(\mathcal{L}(\mathcal{H}))$ by

$$\mathcal{C}_{F, \mathcal{H}} := \left\{ \left(\Gamma(x, y) \left(E^+(x)E^+(y)^* - E^-(x)E^-(y)^* \right) \right) : \Gamma \in \mathbb{K}_X^+(C_b(\Lambda), \mathcal{L}(\mathcal{H})) \right\}.$$

The proof of the first realization theorem relies on the following lemma of independent interest.

Lemma 4.1. *The cone $\mathcal{C}_{F, \mathcal{H}}$ is closed and has non-empty interior. Furthermore, for each $\lambda \in \Lambda$, $1_{\mathcal{L}(\mathcal{H})} \otimes \left(\prod_{\lambda_i \in \lambda} (1 - \psi_i(x)\psi_i(y)^*)^{\lambda_i} \right)_{x, y \in F} \in \mathcal{C}_{F, \mathcal{H}}$.*

Proof. Fix $F \subset X$ finite and a Hilbert space \mathcal{H} , and define the cones \mathcal{C}_F and $\mathcal{C}_{F,\mathcal{H}}$ as above. Following the proof of Lemma 3.4 of [28], we first show that \mathcal{C}_F is closed.

By assumption, for all $x \in X$, there exists $\epsilon_x > 0$ such that $\sup_{\psi \in \Psi} (1 - \psi(x)\psi(x)^*) > \epsilon_x$. Also, for $n := \sup_{\lambda \in \Lambda} |\lambda| < \infty$,

$$E^+(x)E^+(x)^* - E^-(x)E^-(x)^* \geq \epsilon_x^n.$$

Setting $\epsilon = \min_{x \in F} \epsilon_x^n > 0$, we have then that for all $x \in F$, $E^+(x)E^+(x)^* - E^-(x)E^-(x)^* \geq \epsilon$. Therefore, for any $M = \Gamma * (E^+E^{+*} - E^-E^{-*}) \in \mathcal{C}_F$ and any $x \in F$,

$$\|\Gamma(x, x)\| \leq \frac{1}{\epsilon} \max_{x \in F} \|M(x, x)\| \leq \frac{1}{\epsilon} \|M\|.$$

Positivity of Γ then gives $\|\Gamma(x, y)\| \leq \frac{1}{\epsilon} \|M\|$ for all $x, y \in F$. Thus for any Cauchy sequence $(M_n) \subset \mathcal{C}_F$, the corresponding sequence of positive operators (Γ_n) has $(\Gamma_n(x, y))$ in a norm closed ball of $C_b(\Lambda)^*$ and so has a weak-* convergent subsequence. Applying this idea to each pair of points in F , we eventually end up with a subsequence Γ_{ℓ_n} such that for any $x, y \in F$, $\Gamma_{\ell_n}(x, y)$ converges weak-* to $\Gamma(x, y)$. It is not difficult to see that Γ is positive, and so (M_n) converges to some $M = \Gamma * (E^+E^{+*} - E^-E^{-*}) \in \mathcal{C}_F$; that is, \mathcal{C}_F is closed.

Next consider $\mathcal{C}_{F,\mathcal{H}}$. Arguing as above, there exists $\epsilon > 0$ such that for any $M = \Gamma * (E^+E^{+*} - E^-E^{-*}) \in \mathcal{C}_{F,\mathcal{H}}$, $\|\Gamma(x, y)\| \leq \frac{1}{\epsilon} \|M\|$ for all $x, y \in F$. Suppose $(M_n) \subset \mathcal{C}_{F,\mathcal{H}}$ with $\sup_n \|M_n\| = C < \infty$ converging to M . Note that the corresponding sequence (Γ_n) is bounded by C/ϵ . For $h = (h_x) \in \mathcal{H}^{|F|}$ with $\|h\| = 1$, define $M_{h,n}$ by $M_{h,n}(x, y) = \langle M_n(x, y)h_x, h_y \rangle$ and $\Gamma_{h,n}$ by $\Gamma_{h,n}(x, y)(f) = \langle \Gamma_n(x, y)(f)h_x, h_y \rangle$. Then $(M_{h,n}) \subset \mathcal{C}_F$ is a Cauchy sequence, and since \mathcal{C}_F is closed, $\lim_n M_{h,n} = M_h = \Gamma_h * (E^+E^{+*} - E^-E^{-*})$, where $\Gamma_h \geq 0$ and $\|\Gamma_h\| \leq C/\epsilon$. Thus Γ defined via polarization from $\langle \Gamma(f)h, h \rangle = \Gamma_h(f)$ is positive and bounded in norm by C/ϵ , and $M = \Gamma * (E^+E^{+*} - E^-E^{-*})$. Hence the cone $\mathcal{C}_{F,\mathcal{H}}$ is also closed.

We next show that $\mathcal{C}_{F,\mathcal{H}}$ (and as a consequence, \mathcal{C}_F) has non-empty interior. Let $P : X \times X \rightarrow \mathcal{L}(\mathcal{H})$ be a positive kernel with Kolmogorov decomposition $P(x, y) = Q(x)Q(y)^*$. A straightforward argument as in the proof of Lemma 3.5 of [28] shows that the kernel $\Gamma_{P\lambda}$ mapping $X \times X$ to $\mathcal{L}(C_b(\Lambda), \mathcal{L}(\mathcal{H}))$ by

$$\Gamma_{P\lambda}(x, y)(f) = \left((Q(x) \otimes \psi_\lambda^+(x))(Q(y) \otimes \psi_\lambda^+(y))^* - (Q(x) \otimes \psi_\lambda^-(x))(Q(y) \otimes \psi_\lambda^-(y))^* \right)^{-1} f(\lambda)$$

is positive. Thus

$$\left(\Gamma_{P\lambda}(x, y) \left(E^+(x)E^+(y)^* - E^-(x)E^-(y)^* \right) \right) = P(x, y),$$

and so $\mathcal{C}_{F,\mathcal{H}}$ has nonempty interior since it contains all elements of $(\mathcal{L}(\mathcal{H}) \otimes M_{|F|}(\mathbb{C}))^+$.

Finally, the kernel $\Gamma(f) := [1_{\mathcal{L}(\mathcal{H})}]f(\lambda)$ is obviously positive, and

$$\Gamma * (E^+E^{+*} - E^-E^{-*}) = 1_{\mathcal{L}(\mathcal{H})} \otimes (\psi^+ \psi^{+*} - \psi^- \psi^{-*}) = 1_{\mathcal{L}(\mathcal{H})} \otimes \prod_{\lambda_i \in \lambda} (1 - \psi_i \psi_i^*)^{\lambda_i},$$

so restricting to $F \times F$ we have the last statement. \square

We now state and prove our first realization theorem.

Theorem 4.2 (Realization theorem, I). *Let $\varphi : X \rightarrow \mathcal{L}(\mathcal{H})$. The following are equivalent:*

(SC) $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$;

(AD) *There is a positive kernel $\Gamma \in \mathbb{K}_X^+(C_b(\Lambda), \mathcal{L}(\mathcal{H}))$ such that for all $x, y \in X$,*

$$1 - \varphi(x)\varphi(y)^* = \Gamma(x, y) \left(E^+(x)E^+(y)^* - E^-(x)E^-(y)^* \right).$$

In this situation, φ has a transfer function representation.

Proof. Assume that (AD) does not hold. This is equivalent to the statement that for some finite set F , $[1_{\mathcal{L}(\mathcal{H})}] - \varphi\varphi^* \notin \mathcal{C}_{F,\mathcal{H}}$. A Hahn-Banach separation argument gives linear functional $\nu : \mathcal{L}(\mathcal{H}^{|F|}) \rightarrow \mathbb{C}$ such that $\nu(\mathcal{C}_{F,\mathcal{H}}) \geq 0$, $\nu([1_{\mathcal{L}(\mathcal{H})}]) = 1$ and $\nu([1_{\mathcal{L}(\mathcal{H})}] - \varphi\varphi^*) < 0$. Note that $\nu \geq 0$ since $\mathcal{C}_{F,\mathcal{H}} \supseteq (\mathcal{L}(\mathcal{H}) \otimes M_{|F|}(\mathbb{C}))^+$, and so in particular, ν is continuous.

By the Riesz representation theorem, there exists $h = (h(x)) \in \mathcal{H}^{|F|}$ such that $\nu(M) = \langle Mh, h \rangle$. If $F' = \{x \in F : h(x) \neq 0\}$, then $\nu'(M) := \langle Mh|_{F'}, h|_{F'} \rangle$ defines a linear functional separating $[1_{\mathcal{L}(\mathcal{H})}] - \varphi|_{F'}\varphi|_{F'}^*$ from $\mathcal{C}_{F',\mathcal{H}}$. So without loss of generality we assume that for all $x \in F$, $h(x) \neq 0$. We use this to define Hilbert spaces \mathcal{H}_x as the quotient completion of $\mathcal{L}(\mathcal{H})$ under the inner product

$$\langle f(x), g(x) \rangle := \frac{1}{\|h(x)\|^2} \langle f(x)h(x), g(x)h(x) \rangle, \quad f(x), g(x) \in \mathcal{L}(\mathcal{H}).$$

Since $h(x) \neq 0$, \mathcal{H}_x is isomorphic to \mathcal{H} .

on F . Write $1_{\mathcal{F}}$ for the function which equals $1_{\mathcal{L}(\mathcal{H})}$ at every $x \in F$. If $p \in \mathcal{F}$, p^* stands for the element of \mathcal{F} with x th entry $p(x)^*$. Also, let $\chi_x(p)$ denote the element of \mathcal{F} with all entries 0 except the x th, which equals $p(x)$. In this way \mathcal{F} is a unital algebra with addition and multiplication (written as $f \cdot g$) defined entry-wise, and unit $1_{\mathcal{F}}$.

We can also view the (quotient completion of) \mathcal{F} as a Hilbert space $\mathcal{H}_{\mathcal{F}} = \bigoplus_{j=1}^{|F|} \mathcal{H}_j$ with inner product

$$\langle f, g \rangle = \sum_{x \in F} \frac{1}{\|h(x)\|^2} \langle f(x)h(x), g(x)h(x) \rangle.$$

For each $x, y \in F$,

$$\langle k(x, y)f(x), g(y) \rangle = \langle g(y)^*f(x)h(x), h(y) \rangle = \nu((g^*(y)f(x))_{x,y \in F})$$

defines a bounded linear operator $k(x, y) \in \mathcal{L}(\mathcal{H}_x, \mathcal{H}_y)$. For each $x \in F$, identifying \mathcal{H}_x with \mathcal{H} , we have $k = (k(x, y))_{x,y \in F} \in \mathcal{L}(\mathcal{H}^{|F|})$ with

$$\langle kf, g \rangle = \langle (g^*f)h, h \rangle.$$

Extend k to a kernel from $X \times X$ to $\mathcal{L}(\mathcal{H})$ by setting $k(x, y) = 0$ if either x or y is not in F .

Since $\nu \geq 0$ it follows that $k \geq 0$, and so has a Kolmogorov decomposition $k(x, y) = k_y^*k_x$, where $k_x : X \rightarrow \mathcal{E}$ for some Hilbert space \mathcal{E} . We therefore can view $\mathcal{E} \otimes \mathcal{H}_{\mathcal{F}}$ as a Hilbert space with the inner product on elementary tensors given by

$$\langle k_x \otimes f, k_y \otimes g \rangle = \langle k(x, y)f(x), g(y) \rangle.$$

For any $f \in \mathcal{F}$,

$$\begin{aligned} 0 &\leq \nu \left(\left(f(y)^*((1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^+(x))(1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^+(y)^*) - (1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^-(x))(1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^-(y)^*))f(x) \right)_{x,y \in F} \right) \\ &= \left\langle \left(f(y)^*((1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^+(x))(1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^+(y)^*) - (1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^-(x))(1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^-(y)^*))f(x) \right)_{x,y \in F} h, h \right\rangle \\ &= \sum_{x,y} \left\langle ((1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^+(x))(1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^+(y)^*) - (1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^-(x))(1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^-(y)^*))f(x)h(x), f(y)h(y) \right\rangle \\ &= \sum_{x,y} \left\langle ((1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^+(x))k(x, y)(1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^+(y)^*) - (1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^-(x))k(x, y)(1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^-(y)^*))f(x), f(y) \right\rangle \\ &= \left\langle \left(((1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^+)(1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^{+*}) - (1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^-)(1_{\mathcal{L}(\mathcal{H})} \otimes \psi_{\lambda}^{-*})) * k \right) f, f \right\rangle. \end{aligned}$$

Since $k(x, y)$ is 0 when x or y is not in F , this suffices to show that k is an admissible kernel.

The calculation

$$\begin{aligned}
0 &\geq v \left((1_{\mathcal{L}(\mathcal{H})} - \varphi(x)\varphi(y)^*)_{x,y \in F} \right) \\
&= \left\langle (1_{\mathcal{L}(\mathcal{H})} - \varphi(x)\varphi(y)^*)_{x,y \in F} h, h \right\rangle \\
&= \sum_{x,y} [\langle h(x), h(y) \rangle - \langle \varphi(x)\varphi(y)^* h(x), h(y) \rangle] \\
&= \sum_{x,y} \langle (k(x,y) - \varphi(x)k(x,y)\varphi(y)^*) 1_{\mathcal{L}(\mathcal{H})}, 1_{\mathcal{L}(\mathcal{H})} \rangle \\
&= \langle ([1_{\mathcal{L}(\mathcal{H})}] - \varphi\varphi^*) * k, 1_{\mathcal{F}}, 1_{\mathcal{F}} \rangle,
\end{aligned} \tag{4.1}$$

then implies that $\varphi \notin H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$.

So far we have shown that $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ implies the Agler decomposition holds when restricted to any finite set F . A standard application of Kurosh's theorem (see, for example, [28]) then gives the existence of the Agler decomposition on the whole of X .

Now suppose that $\varphi : X \rightarrow \mathcal{L}(\mathcal{H})$ and that (AD) holds; that is there is a positive kernel $\Gamma \in \mathbb{K}_X^+(C_b(\Lambda), \mathcal{L}(\mathcal{H}))$ such that for all $x, y \in X$,

$$1 - \varphi(x)\varphi(y)^* = \Gamma(x, y) (E^+(x)E^+(y)^* - E^-(x)E^-(y)^*).$$

Fix a finite set $F \subset X$ and an admissible kernel $k \in \mathcal{K}_{\Lambda, \mathcal{H}}$. Then on $F \times F$,

$$\begin{aligned}
&(1 - \varphi(x)^*\varphi(y)) * (k(x, y)) \\
&= (\Gamma(x, y) * (E^+(x)E^+(y)^* - E^-(x)E^-(y)^*)) * (k(x, y)) \\
&= (\gamma(x)(Z^+(x)Z^+(y)^* - Z^-(x)Z^-(y)^*)\gamma(y)^*) * (k(x, y)) \\
&= \left(\gamma(x) \sum_{\lambda \in \Lambda} P_\lambda \otimes (E^+(x)(\lambda)E^+(x)(\lambda)^* - E^-(x)(\lambda)E^-(x)(\lambda)^*) \gamma(y)^* \right) * (k(x, y)) \\
&= \left(\gamma(x) \sum_{\lambda \in \Lambda} P_\lambda \otimes (\psi_\lambda^+(x)k(x, y)\psi_\lambda^+(x)^* - \psi_\lambda^-(x)k(x, y)\psi_\lambda^-(x)^*) \gamma(y)^* \right),
\end{aligned}$$

which is positive, since for each $\lambda \in \Lambda$,

$$(\psi_\lambda^+(x)k(x, y)\psi_\lambda^+(x)^* - \psi_\lambda^-(x)k(x, y)\psi_\lambda^-(x)^*) = (\psi_\lambda^+\psi_\lambda^{+*} - \psi_\lambda^-\psi_\lambda^{-*}) * k \geq 0.$$

Thus $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, and so (SC) and (AD) are equivalent.

Assuming (AD), we show that φ has a transfer function representation by employing a standard lurking isometry argument. To begin with, we have a Kolmogorov decomposition $\Gamma = \gamma^*\gamma$, and by Proposition 2.1, for all $\lambda \in \Lambda$, the entries of Z^\pm satisfy $Z_\lambda^\pm(x) = \rho(E_\lambda^\pm(x))$ entry-wise (that is, for all λ). Hence

$$1 - \varphi(x)\varphi(y)^* = \gamma(x)Z^+(x)Z^+(y)^*\gamma(y)^* - \gamma(x)Z^-(x)Z^-(y)^*\gamma(y)^*, \tag{4.2}$$

and so bringing negative terms over to opposite sides of the equation, we have by the usual arguments the existence of a unitary $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{L}(\mathcal{E} \oplus \mathcal{H})$ such that

$$\begin{pmatrix} Z^+(x)^*\gamma(x)^* \\ \varphi(x)^* \end{pmatrix} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} Z^-(x)^*\gamma(x)^* \\ 1 \end{pmatrix} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} S(x)^*Z^+(x)^*\gamma(x)^* \\ 1 \end{pmatrix},$$

where $S(x) = Y(x)Z^-(x)$. Hence $C = \gamma(x)Z^+(x)(1 - S(x)A)$, and so

$$\gamma(x) = C(1 - S(x)A)^{-1}Y(x). \tag{4.3}$$

Plugging this into the second equation,

$$\begin{aligned}
\varphi(x) &= D + \gamma(x)Z^+(x)S(x)B \\
&= D + C(1 - S(x)A)^{-1}Y(x)Z^+(x)S(x)B \\
&= D + C(1 - S(x)A)^{-1}P(x)S(x)B \\
&= D + C(1 - S(x)A)^{-1}S(x)B \\
&= D + CS(x)(1 - AS(x))^{-1}B;
\end{aligned}$$

that is, φ has a transfer function representation. \square

4.2. Realizations for ample preorderings. The realization theorem Theorem 4.2 has the drawback that having a transfer function representation is not enough to ensure that a function is in $H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. There are circumstances in which this difficulty can be circumvented. For example, if Ψ contains d test functions over a set X and $\Lambda = \{e_j\}_{j=1}^d$ (which is the classical setting), we have the result presented in Theorem 2.13.

The reason we get so much more with the classical realization theorems is that the auxiliary test functions are the same as the test functions and these are by construction in our algebra. As it happens, with ample preorderings, something similar occurs (Theorem 2.19). One consequence of the next theorem is that in the setting of ample preorderings, $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ inherits its norm from the transfer function algebra $\mathcal{T}(X, \Lambda, \mathcal{H})$, and in fact the two are equal, thus strengthening Lemma 3.10 in this context.

Theorem 4.3 (Realization theorem, II). *Suppose $\Psi = \{\psi_1, \dots, \psi_d\}$ is a collection of test functions and Λ is an ample preordering. The following are equivalent:*

- (SC) $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$;
- (AD1) *There exist positive kernels $\Gamma_\lambda : X \times X \rightarrow \mathcal{L}(\mathcal{H})$ so that for all $x, y \in X$*

$$1_{\mathcal{H}} - \varphi(x)\varphi(y)^* = \sum_{\lambda \in \Lambda} \Gamma_\lambda(x, y) \prod_{\ell_j \in \lambda} ([1] - \psi_{\ell_j}(x)\psi_{\ell_j}(y)^*)$$
- (AD2) *There exist positive kernels $\tilde{\Gamma}_\lambda : X \times X \rightarrow \mathcal{L}(\mathbb{C}^{|\lambda|}, \mathcal{H})$ so that for all $x, y \in X$*

$$1_{\mathcal{H}} - \varphi(x)\varphi(y)^* = \sum_{\lambda \in \Lambda} \tilde{\Gamma}_\lambda(x, y) ([1_{\mathbb{C}^{|\lambda|}}] - \sigma_\lambda(x)\sigma_\lambda(y)^*)$$
- (TF) *There is a colligation $\Sigma = (U, \rho, \mathcal{E})$ so that $\varphi = W_\Sigma$;*
- (vN-a) *For every representation of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ which is strictly contractive on auxiliary test functions or which is contractive on auxiliary test functions and either strongly or weakly continuous, $\|\pi(\varphi)\| \leq 1$.*

Proof. The proof that (SC) \Leftrightarrow (AD1) follows directly from Theorem 4.2. A straightforward factorization shows that (AD1) \Rightarrow (AD2). The standard lurking isometry argument as in that theorem then gives (AD2) \Rightarrow (TF). That the weak form of (vN-a) implies the strong form which then implies the strict form is also immediate.

By Theorem 2.19, the auxiliary test functions are in $H_1^\infty(\mathcal{K}_{\Lambda, \mathbb{C}^n})$ for appropriate n and these functions generates the same collection of admissible kernels. Using the fact that the operator in the colligation for φ is unitary, the usual sort of calculation shows that if φ has a transfer function representation, then for $G = C(1 - SA)^{-1}$,

$$((1 - \varphi(x)\varphi(y)^*)k(x, y)) = (G(x)((1 - S(x)S(y)^*)k(x, y))G(y)^*) \geq 0,$$

and so (TF) \Rightarrow (SC).

If π is a representation of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ which is strictly contractive on auxiliary test functions, and if we interpret

$$\pi(\varphi) = D \otimes 1 + C \otimes \pi(S)(1 \otimes 1 - A \otimes \pi(S))^{-1} B \otimes 1, \quad \pi(S) = \sum_{\lambda \in \Lambda} P_\lambda \otimes \pi(\sigma_\lambda),$$

then a nearly identical argument to that of the last paragraph shows that $1 - \pi(\varphi)\pi(\varphi)^* \geq 0$; that is, π is a contractive representation of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. Hence (TF) implies the strict form of (VN-a). On the other hand, if π is only assumed to be weakly continuous, then we argue as in [28], first scaling A and C to rA and rC for $r < 1$ and calling the resulting functions φ_r , then approximating φ_r by polynomials in S as at the end of the proof Theorem 3.3. The representation is easily seen to be contractive on these polynomials. Since these can be chosen to converge pointwise to φ_r , the representation is contractive on φ_r for all r . Taking r to 1 we have pointwise convergence to φ , and so once again, $1 - \pi(\varphi)\pi(\varphi)^* \geq 0$.

Finally, suppose that the strict form of (vN-a) holds. Fix $\varphi \in H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. We show that for $k \in \mathcal{K}_{\Lambda, \mathcal{H}}$, $([1_{\mathcal{H}}] - \varphi\varphi^*) * k \geq 0$, and so (SC) holds as well. For this, it suffices to prove that for fixed $k \in \mathcal{K}_{\Lambda, \mathcal{H}}$, $([1_{\mathcal{H}}] - \varphi\varphi^*) * k \geq 0$ when we restrict to a finite subset $F \subset X$. So fix a finite set $F \subset X$. On F replace the test functions Ψ by $\Psi_r = \{\psi_r = r\psi : \psi \in \Psi\}$, where $r > 1$ is sufficiently close to 1 so that $\sup_{\psi_r \in \Psi_r} |\psi_r(x)| < 1$ for all $x \in F$ (this is possible since F is finite). Define in the same way as before, $\mathcal{K}_{\Lambda, \mathcal{H}}^r$ on F with these test functions, as well as $H^\infty(F, \mathcal{K}_{\Lambda, \mathcal{H}}^r)$.

Since for $k_r \in \mathcal{K}_{\Lambda, \mathcal{H}}^r$ and $\lambda \in \Lambda$,

$$\prod_{\ell \in \lambda} ([1_{\mathcal{H}}] - \psi_\ell \psi_\ell^*) * k_r = \frac{1}{r^2} \left((r^2 - 1)[1_{\mathcal{H}}] + [1_{\mathcal{H}}] - \psi_{r,\ell} \psi_{r,\ell}^* \right) * k_r \geq 0, \quad (4.4)$$

so it follows that $\mathcal{K}_{\Lambda, \mathcal{H}}^r \subseteq \mathcal{K}_{\Lambda, \mathcal{H}}$ on F . Hence $H^\infty(F, \mathcal{K}_{\Lambda, \mathcal{H}}) := H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})|_F \subseteq H^\infty(F, \mathcal{K}_{\Lambda, \mathcal{H}}^r)$. Hence, we view $H^\infty(F, \mathcal{K}_{\Lambda, \mathcal{H}})$ as a subalgebra of $H^\infty(F, \mathcal{K}_{\Lambda, \mathcal{H}}^r)$ endowed with the $H^\infty(F, \mathcal{K}_{\Lambda, \mathcal{H}}^r)$ -norm, which we write as $\|\cdot\|_r$. For $k_r \in \mathcal{K}_{\Lambda, \mathcal{H}}^r$, the map π taking $f \in H^\infty(F, \mathcal{K}_{\Lambda, \mathcal{H}})$ to M_f on $H^2(k_r)$ defines a strictly contractive representation of $H^\infty(F, \mathcal{K}_{\Lambda, \mathcal{H}})$, since (4.4) implies that for $\psi \in \Psi$, $\|\psi\|_r \leq 1/r$. It follows then that under the assumption that the strict form of (vN-a) holds for φ ,

$$([1_{\mathcal{H}}] - \varphi^* \varphi) * k_r \geq 0, \quad k_r \in \mathcal{K}_{\Lambda, \mathcal{H}}^r. \quad (4.5)$$

Fix $k \in \mathcal{K}_{\Lambda, \mathcal{H}}$. For any $r > 1$ as above, the kernel k_0 defined by

$$k_0(x, y) = \begin{cases} k(x, x) & y = x; \\ 0 & \text{otherwise,} \end{cases}$$

defines a kernel in $\mathcal{K}_{\Lambda, \mathcal{H}}^r$. Fix $t \in (0, 1)$. We show that $tk - (1 - t)k_0 \in \mathcal{K}_{\Lambda, \mathcal{H}}^r$ for any $r > 1$ and sufficiently close to 1. It will follow then that for such r and for all $\lambda \in \Lambda$,

$$\begin{aligned} & \prod_{\ell \in \lambda} ([1] - \psi_{r,\ell} \psi_{r,\ell}^*) * [tk - (1 - t)k_0] \\ &= \prod_{\ell \in \lambda} \left[t([1] - \psi_\ell^* \psi_\ell) * k + (1 - t)([1] - \psi_{r,\ell} \psi_{r,\ell}^*) * k_0 - t(r^2 - 1)\psi_\ell \psi_\ell^* * k \right] \geq 0. \end{aligned} \quad (4.6)$$

We do this by proving that in this case,

$$(1 - t) \prod_{\ell \in \lambda} ([1] - \psi_{r,\ell} \psi_{r,\ell}^*) * k_0 \geq t(r^2 - 1) \prod_{\ell \in \lambda} \psi_\ell \psi_\ell^* * k.$$

Assume that $k(x, x) > 0$ for all $x \in F$. We can do this since if $k(x, x) = 0$ for some x , then $k(x, y) = k(y, x) = 0$ for all y . So without loss of generality we restrict to $F' \subset F$ with the property that the diagonal entries of k are strictly positive. Since F is finite $(1 - t) \prod_{\ell \in \lambda} ([1] - \psi_{r,\ell}^* \psi_{r,\ell}) * k_0$ is a

diagonal matrix with diagonal entries of the form $(1-t) \prod_{\ell \in \lambda} (1-r^2 |\psi_\ell(x)|) k(x, x) \geq (1-t) \prod_{\ell \in \lambda} (1-r^2 |\psi_\ell(x)|) \epsilon > 0$ for some $\epsilon > 0$.

Hence it suffices to show that for 1_F the usual identity matrix over $\mathbb{C}^{|F|}$,

$$\frac{\epsilon(1-t)}{t} \prod_{\ell \in \lambda} \frac{1-r^2 |\psi_\ell(x)|}{r^2-1} 1_F \otimes 1_{\mathcal{H}} \geq \prod_{\ell \in \lambda} \psi_\ell \psi_\ell^* k.$$

However, as $r \searrow 1$, $\frac{1-r^2 |\psi_\ell(x)|}{r^2-1} \nearrow \infty$. Thus for r sufficiently close to 1, (4.6) is satisfied.

Since (4.5) holds for all sufficiently small $r > 1$ and $k_r \in \mathcal{K}_{\Lambda, \mathcal{H}}^r$, it follows that for all such r and $k \in \mathcal{K}_{\Lambda, \mathcal{H}}^r$ and $t \in (0, 1)$,

$$([1_{\mathcal{H}}] - \varphi \varphi^*) * [tk + (1-t)k_0] \geq 0,$$

and so taking $t \nearrow 1$, we have $([1_{\mathcal{H}}] - \varphi \varphi^*) * k \geq 0$ on $F \times F$. The set $F \subset X$ was arbitrary, and so we conclude that $\varphi \in H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. \square

Corollary 4.4. *With Λ an ample preordering, $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ is isometrically isomorphic to $\mathcal{T}(X, \Lambda, \mathcal{H})$ and $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ is isometrically isomorphic to $\mathcal{T}^A(X, \Lambda, \mathcal{H})$.*

Corollary 4.5. *Let Λ be an ample preordering over d test functions. Then any strictly contractive or weakly continuous contractive representation of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ which is 2^{d-1} -contractive is completely contractive. Likewise, any 2^{d-1} -contractive representation of $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ is completely contractive.*

Proof. This follows from the last two theorems since a representation which is 2^{d-1} -contractive is contractive on auxiliary test functions. \square

4.3. Agler-Pick interpolation. It is now standard practice to apply the realization theorem to Pick type interpolation problems.

Theorem 4.6 (Agler-Pick interpolation). *Let Λ be an ample preordering, $X_0 \subseteq X$. For each $x \in X_0$, let $a_x, b_x \in \mathcal{L}(\mathcal{H})$. The following are equivalent:*

- (i) *There exists $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ such that for all $x \in X_0$, $b_x = \varphi(x)a_x$;*
- (ii) *There exists a positive kernel $\Gamma : X_0 \times X_0 \rightarrow C_b(\Lambda, \mathcal{L}(\mathcal{H}))$ so that for all $x, y \in X_0$*

$$\left(((a_x a_y^* - b_x b_y^*) \otimes 1_n) * k_\sigma(x, y) \right) \geq 0,$$

where $k_\sigma = 1_{\mathcal{L}(\mathcal{H})} \otimes (1 - \sigma_{\lambda^m}(x) \sigma_{\lambda^m}(y)^)^{-1}$ and $n = 2^{|\lambda|-1}$.*

Proof. The proof follows the first part of the proof of the realization theorem, giving a transfer function W such that $b_x = W(x)a_x$ for all $x \in X_0$. This transfer function is well defined for all $x \in X$, and hence W extends to $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. \square

In the case of the bidisk, the function σ will be a 2×2 matrix valued inner function, and though heavily constrained, it will not be uniquely determined. For practical purposes it would be very interesting to know a concrete example of a choice of σ .

Taking $b_x = \sqrt{\delta}$ for all $x \in X_0 = X$ in Theorem 4.6 gives the so-called *Toeplitz-corona theorem*. We need a special case of this, stated in the following lemma.

Lemma 4.7. *Let Λ be an ample preordering. For $\lambda \in \Lambda$, there exist ω_λ with entries in $H^\infty(X, \mathcal{K}_\Lambda)$ such that for all $x \in X$, $\psi_\lambda^+(x) \omega_\lambda(x) = 1$. Consequently, given a unitary colligation (U, \mathcal{E}, ρ) , there exists Y with entries in $H^\infty(X, \mathcal{K}_\Lambda)$ such that for all x , $Z^+(x)Y(x) = 1$.*

Proof. Recall from the definition in subsection 2.7, $\psi_\lambda^+ : X \rightarrow \mathbb{C}^{2^{|\lambda|-1}}$ with entries which are products of test functions and hence in $H_1^\infty(X, \mathcal{K}_\Lambda)$ and the first entry equal to 1. If we replace this 1 by 0 and call the resulting function $\tilde{\psi}_\lambda^+$, then we see that for any admissible kernel k ,

$$(\psi_\lambda^+ \psi_\lambda^{+*} - [1]) * k_\sigma = (\tilde{\psi}_\lambda^+ \tilde{\psi}_\lambda^{+*}) * k_\sigma \geq 0,$$

where k_σ is as in the statement of Theorem 4.6.

Now apply Theorem 4.6 to get a function in $\omega_\lambda \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathbb{C}^{2^{|\lambda|-1}}})$. It might be objected that the ψ_λ^\pm s are not square matrices, but this can be rectified by padding with rows of zeros. The definition of Y in terms of the ω_λ s then yields the final statement of the theorem. \square

Corollary 4.8. *Let Λ be an ample preordering, (U, \mathcal{E}, ρ) a unitary colligation. Then for S and Y chosen as in Theorem 2.19 and Lemma 4.7,*

$$\gamma(x) := C(1 - S(x)A)^{-1}Y(x)$$

is the pointwise limit functions in $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$.

Proof. By the now standard arguments whereby we scale A and C to rA and rC , $0 < r < 1$, approximate $rC(1 - S(x)rA)^{-1}$ by polynomials in S and then take limits, the result follows since the entries of S and Y are in $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. \square

4.4. Brehmer representations again. Using the last corollary, we can now include a statement concerning Brehmer representations to the realization theorem for ample preorderings.

Theorem 4.9 (Realization theorem, III). *Suppose $\Psi = \{\psi_1, \dots, \psi_d\}$ is a collection of test functions and Λ is an ample preordering. The following are equivalent:*

(SC) $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$;

(AD1) *There exist positive kernels $\Gamma_\lambda : X \times X \rightarrow \mathcal{L}(\mathcal{H})$ so that for all $x, y \in X$*

$$1 - \varphi(x)\varphi(y)^* = \sum_{\lambda \in \Lambda} \Gamma_\lambda(x, y) \prod_{\ell_j \in \lambda} ([1] - \psi_{\ell_j}(x)\psi_{\ell_j}(y)^*)$$

(AD2) *There exist positive kernels $\tilde{\Gamma}_\lambda : X \times X \rightarrow \mathcal{L}(\mathbb{C}^{|\lambda|}, \mathcal{H})$ so that for all $x, y \in X$*

$$1 - \varphi(x)\varphi(y)^* = \sum_{\lambda \in \Lambda} \tilde{\Gamma}_\lambda(x, y) ([1_{\mathbb{C}^{|\lambda|}}] - \sigma_\lambda(x)\sigma_\lambda(y)^*)$$

(TF) *There is a unitary colligation $\Sigma = (U, \rho, \mathcal{E})$ so that $\varphi = W_\Sigma$;*

(vN-B) *For every strict / strongly continuous / weakly continuous Brehmer representation π of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, $\|\pi(\varphi)\| \leq 1$;*

(vN-a) *For every representation of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ which is strictly contractive on auxiliary test functions or which is contractive on auxiliary test functions and either strongly or weakly continuous, $\|\pi(\varphi)\| \leq 1$.*

Proof. The equivalence of all parts except for (vN-B) are the content of Theorem 4.3. Since by Lemma 3.10, representations which are strictly contractive on auxiliary test functions are strictly contractive Brehmer representations, to finish the proof it suffices to prove that if (AD1) holds, then any weakly contractive Brehmer representation is contractive.

We can rewrite the statement of (AD1) as being that there exists a positive kernel Γ with Kolmogorov decomposition $\Gamma = \gamma\gamma^*$ such that for all x, y ,

$$1 - \varphi(x)\varphi(y)^* = \gamma(x)(Z^+(x)Z^{+*}(y) - Z^-(x)Z^{-*}(y))\gamma^*.$$

A lurking isometry argument then gives that there is a unitary operator $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, so that

$$C = \gamma(x)(Z^+(x) - Z^-(x)A).$$

According to Lemma 4.7, we can choose S with entries in $H^\infty(X, \mathcal{K}_\Lambda)$ to be strictly contractive for all x , and so $\gamma(x)Z^+(x) = C(1 - S(x)A)^{-1}$. Applying Lemma 4.7, we can choose Y with entries in $H^\infty(X, \mathcal{K}_\Lambda)$ such that for all x , $Z^+(x)Y(x) = 1$, and hence

$$\gamma(x) = C(1 - S(x)A)^{-1}Y(x),$$

which by Corollary 4.8 as a limit of functions in $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. The lurking isometry argument also gives that

$$\varphi(x) = D + \gamma(x)Z^-(x)B.$$

Let π be a weakly continuous Brehmer representation. If γ has entries in $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$, then

$$\pi(\varphi) = 1 \otimes D + (\pi(\gamma)\pi(Z^-)) \otimes B,$$

where we are using the shorthand notation of “ $\pi(\gamma)$ ” and “ $\pi(Z^-)$ ” for the entrywise application of π to these functions. A straightforward calculation using the fact that U in the colligation is unitary gives

$$1 - \pi(\varphi)\pi(\varphi)^* = \pi(\gamma)(\pi(Z^+)\pi(Z^+)^* - \pi(Z^-)\pi(Z^-)^*)\pi(\gamma)^* \geq 0.$$

More generally, we approximate γ by function with entries in $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$. Taking limits, we still find that $1 - \pi(\varphi)\pi(\varphi)^* \geq 0$; that is, π is contractive. \square

We close this section with an analogue of Brehmer’s theorem.

Proposition 4.10. *Let π be a Brehmer representation of $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ or weakly continuous Brehmer representation of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}}) \subset C_b(X, \mathcal{L}(\mathcal{H}))$. Then π dilates to a (weakly continuous) Brehmer *-representation $\tilde{\pi}$ of $C_b(X, \mathcal{L}(\mathcal{H}))$.*

Proof. This follows from Corollary 4.4 and Theorem 3.11. \square

4.5. Algebras generated by two test functions. Brehmer’s original theorem is a dilation theorem which works over the polydisk, but requires a special class of representations. On the other hand, for the \mathbb{D}^2 , Andô’s theorem shows that such dilation results hold for a broader class of representations. We first state a bidisk version of the realization theorem. The emphasis here is on the equivalence of the two versions of von Neumann’s inequality, since by Lemma 3.10, the collection of representations which are strictly contractive on auxiliary test functions is the smallest set of representations we consider, while the weakly continuous Brehmer representations form the largest set.

Theorem 4.11. *Let $\varphi : \mathbb{D}^2 \rightarrow \mathcal{L}(\mathcal{H})$. The following are equivalent:*

- (SC) $([1] - \varphi\varphi^*) * k_s \geq 0$, $k_s(z, w) = (1 - z_1 w_1)^{-1}(1 - z_2 w_2)^{-1}$, or equivalently $\varphi \in H_1^\infty(\mathbb{D}^2, \mathcal{L}(\mathcal{H}))$;
- (AD1) For every admissible kernel $k \in \{k \geq 0 : (1 - Z_j Z_j^*) * k \geq 0, j = 1, 2\}$, we have $(1 - \varphi\varphi^*) * k \geq 0$;
- (AD2) There exist positive kernels Γ_1, Γ_2 such that $[1] - \varphi\varphi^* = \Gamma_1 * ([1] - Z_1 Z_1^*) + \Gamma_2 * ([1] - Z_2 Z_2^*)$, where $Z_j(z) = z_j$;
- (vN1) $\varphi \in H^\infty(\mathbb{D}^2, \mathcal{L}(\mathcal{H}))$ and for every weakly continuous Brehmer representation π (so $1 - \pi(z_1)\pi(z_1)^* - \pi(z_2)\pi(z_2)^* + \pi(z_1)\pi(z_2)\pi(z_1)^*\pi(z_2)^* \geq 0$), we have $\|\pi(\varphi)\| \leq 1$;
- (vN2) $\varphi \in H^\infty(\mathbb{D}^2, \mathcal{L}(\mathcal{H}))$ and for every strictly contractive representation π (so $\|\pi(z_i)\| < 1$), we have $\|\pi(\varphi)\| \leq 1$.

We next show how to generalize this to algebras over general domains generated by a pair of test functions.

Let us assume that $\Psi = \{\psi_1, \psi_2\}$ is a collection of test functions on a set X and Λ be the standard ample preordering with maximal element $(1, 1)$, while $\Lambda_0 = \{(1, 0), (0, 1)\}$, the nearly ample preordering used for the standard realization theorem. Write \mathcal{K}_0 for the set of admissible kernels associated to Λ_0 ; so $k \in \mathcal{K}_0$ means that $k \geq 0$ and $([1] - \psi_j \psi_j^*) * k \geq 0$, $j = 1, 2$.

By assumption, for each $x \in X$, $\max\{|\psi_1(x)|, |\psi_2(x)|\} < 1$ and the elements of Ψ separate the points of X . Hence by Lemma 2.17, there is an injective mapping ξ of X onto a subset Ω of \mathbb{D}^2 given by $\xi : x \mapsto z = (z_1, z_2) = (\psi_1(x), \psi_2(x))$.

Theorem 4.12. *Let $\Psi = \{\psi_1, \psi_2\}$ be a collection of test functions on a set X . Let Λ be the ample preordering with maximal element $(1, 1)$, Λ_0 the preordering $\{(1, 0), (0, 1)\}$. For $\varphi : \mathbb{D}^2 \rightarrow \mathcal{L}(\mathcal{H})$, the following are equivalent:*

- (SC1) $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{L}(\mathcal{H})})$, the closed unit ball of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{L}(\mathcal{H})})$;
- (SC2) $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda_0, \mathcal{L}(\mathcal{H})})$, the closed unit ball of $H^\infty(X, \mathcal{K}_{\Lambda_0, \mathcal{L}(\mathcal{H})})$;
- (vN1) $\varphi \in H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{L}(\mathcal{H})})$ and for every weakly continuous Brehmer representation π , we have $\|\pi(\varphi)\| \leq 1$;
- (vN2) $\varphi \in H^\infty(X, \mathcal{K}_{\Lambda_0, \mathcal{L}(\mathcal{H})})$ and for every strictly contractive representation π (so $\|\pi(\psi_i)\| < 1$), we have $\|\pi(\varphi)\| \leq 1$.

In particular, Λ and Λ_0 are equivalent preorderings; that is, $H^\infty(X, \mathcal{K}_{\Lambda_0, \mathcal{L}(\mathcal{H})}) = H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{L}(\mathcal{H})})$.

Proof. The implication (SC2) implies (SC1) is trivial, while (vN1) is equivalent to (SC1) and (vN2) is equivalent to (SC2) by Theorem 4.3. We therefore only need to show that (SC1) implies (SC2).

Assume $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{L}(\mathcal{H})})$. Using the embedding of X in the polydisk given in Lemma 2.17, we let $\tilde{\varphi} = \varphi \circ \xi^{-1}$. Fix $\lambda = (1, 1)$. We suppress the dependence on λ in the following, writing $\psi^+ = \begin{pmatrix} 1 & \psi_1 \psi_2 \end{pmatrix}$, $\psi^- = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}$, and $\sigma_0 = \psi^{+*} |\psi^+|^{-2} \psi^-$. When composed with ξ^{-1} these become $\tilde{\psi}^+ = \begin{pmatrix} 1 & z_1 z_2 \end{pmatrix}$, $\tilde{\psi}^- = \begin{pmatrix} z_1 & z_2 \end{pmatrix}$, and $\tilde{\sigma}_0 = \tilde{\psi}^{+*} |\tilde{\psi}^+|^{-2} \tilde{\psi}^-$ on $\Omega = \xi(X) \subset \mathbb{D}^2$.

By Theorem 2.19, $\tilde{\sigma}_0$ extends to an auxiliary test function $\tilde{\sigma} \in H^\infty(\mathbb{D}^2, M_2(\mathbb{C}))$ corresponding to λ . Set $\sigma = (\tilde{\sigma}|\Omega) \circ \xi$. Easy calculations show that $\psi^+ \sigma = \psi^-$ and $([1_2] - \sigma \sigma^*) * (k_s \otimes 1_2) \geq 0$, where $k_s(x, y) = (1 - \psi_1(x) \psi_1(y))^{-1} (1 - \psi_2(x) \psi_2(y))^{-1}$. In other words, σ is an auxiliary test function corresponding to $\lambda = (1, 1)$, this time over X .

Also by Theorem 2.19, $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda, \mathcal{L}(\mathcal{H})})$ if and only if $(1_2 \otimes ([1_{\mathcal{L}(\mathcal{H})}] - \varphi \varphi^*)) * k_\sigma \otimes 1_{\mathcal{L}(\mathcal{H})} \geq 0$, where $k_\sigma(x, y) = (1_2 - \sigma(x) \sigma(y))^{-1}$. Translating over to Ω , this becomes

$$(1_2 \otimes ([1_{\mathcal{L}(\mathcal{H})}] - \tilde{\varphi} \tilde{\varphi}^*)) * k_{\tilde{\sigma}} \otimes 1_{\mathcal{L}(\mathcal{H})} \geq 0,$$

on $\Omega \times \Omega$, where $k_{\tilde{\sigma}}$ is defined analogously to k_σ . By Theorem 4.6, $\tilde{\varphi}$ extends to a function in $H^\infty(\mathbb{D}^2, \mathcal{L}(\mathcal{H}))$, and so by Agler's realization theorem in the case of the bidisk (Theorem 2.7), $\tilde{\varphi} \in H_1^\infty(\mathbb{D}^2, \mathcal{K}_{\Lambda_0, \mathcal{L}(\mathcal{H})})$. Restricting to Ω and mapping back to X , it follows that $\varphi \in H_1^\infty(X, \mathcal{K}_{\Lambda_0, \mathcal{L}(\mathcal{H})})$. \square

4.6. Realizations with nearly ample preorderings. It is now possible to extend the results of the previous section to more than two test functions. Following the template set there, we first do this over the polydisk, thus obtaining a generalization of the results in [30], and then to general algebras obtained with a finite collection of test functions. We begin with a d -variable version of Theorem 4.11.

Throughout this section we assume that we have the standard ample preordering $\Lambda^a = \{1\}$ over the collection of test functions is $\Psi = \{\psi_1, \dots, \psi_d\}$, where here 1 stands for the d -tuple with all values 1, and a standard nearly ample preordering $\Lambda^{na} = \{\lambda_1, \lambda_2\}$, where λ_i is a d -tuple with the

j_i th entry equal to 0 and all others equal to 1, and $\lambda_1 \neq \lambda_2$. In the first case, the collection of kernels is particularly simple. By Lemma 2.4 they are all subordinate to the so-called Szegő kernel, k_s . In the nearly ample case the set is more complex, since then

$$k \in \mathcal{K}_{\Lambda^{na}} := \{k \geq 0 : \prod_{j \neq j_1} (1 - \psi_j \psi_j^*) * k \geq 0 \text{ and } \prod_{j \neq j_1} (1 - Z_j Z_j^*) * k \geq 0, j = 1, 2\}.$$

Recall from Theorem 2.12 that over the polydisk with test functions equal to the coordinate functions, the algebras we get from these two collections of kernels are the same, with equal norms. Also, since by Lemma 2.2 $\{1, \lambda_1, \lambda_2\}$ is also an ample preordering equivalent to Λ^a , by Theorem 2.19, for any collection of d test functions over a set X , we have that the auxiliary test functions $\sigma_1 \in H^\infty(X, \mathcal{K}_{\Lambda, \mathbb{C}^{2d-1}})$ and $\sigma_{\lambda_1}, \sigma_{\lambda_2} \in H^\infty(X, \mathcal{K}_{\Lambda, \mathbb{C}^{2d-2}})$. Thus we have the following generalization of the main theorem of [30].

Theorem 4.13. *Let $\varphi : \mathbb{D}^d \rightarrow \mathcal{L}(\mathcal{H})$. The following are equivalent:*

- (SC1) $([1] - \varphi \varphi^*) * k_s \geq 0$, or equivalently, $\varphi \in H_1^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$;
- (SC2) For every admissible kernel $k \in \mathcal{K}_{\Lambda^{na}}$, $([1] - \varphi \varphi^*) * k \geq 0$;
- (AD) There exists a positive kernels $\Gamma, \Gamma_1, \Gamma_2$ such that

$$[1] - \varphi \varphi^* = \Gamma * \prod_{j=1}^d ([1] - Z_j Z_j^*) = \Gamma_1 * \prod_{j \neq j_1} ([1] - Z_j Z_j^*) + \Gamma_2 * \prod_{j \neq j_2} ([1] - Z_j Z_j^*),$$

where $Z_j(z) = z_j$;

- (TF) There are unitary colligations Σ_a and Σ_{na} in the ample and nearly ample setting respectively, such that φ has transfer function representations

$$\varphi = W_{\Sigma_a} = W_{\Sigma_{na}};$$

- (vN-a) $\varphi \in H^\infty(\mathbb{D}^d, \mathcal{H}) = H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^a, \mathcal{H}}) = H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^{na}, \mathcal{H}})$ and for every representation π which is strictly contractive on the auxiliary test function σ_1 (respectively, the auxiliary test functions $\sigma_{\lambda_1}, \sigma_{\lambda_2}$), or contractive on these and either strongly or weakly continuous, we have $\|\pi(\varphi)\| \leq 1$;
- (vN-B) $\varphi \in H^\infty(\mathbb{D}^d, \mathcal{H}) = H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^a, \mathcal{H}}) = H^\infty(\mathbb{D}^d, \mathcal{K}_{\Lambda^{na}, \mathcal{H}})$ and for every representation π which is a strict / strongly continuous / weakly continuous Brehmer representation with respect to Λ^a (respectively, Λ^{na}), we have $\|\pi(\varphi)\| \leq 1$.

There are also statements regarding transfer function representations which we have not included here.

Proof of Theorem 4.13. This follows immediately from Theorem 2.12 and an application of the realization theorem to the two equivalent preorderings Λ^a and Λ^{na} . \square

We now state a d -variable version of Theorem 4.12. As usual, k_s stands for the Szegő kernel $\prod_1^d ([1] - \psi_j \psi_j^*)^{-1}$.

Theorem 4.14 (Realization theorem, IV). *Suppose $\Psi = \{\psi_1, \dots, \psi_d\}$ is a collection of test functions over a set X , Λ^{na} is a standard nearly ample preordering under the standard ample preordering $\Lambda^a = \{1\}$. The following are equivalent:*

- (SC1) $([1] - \varphi \varphi^*) * k_s \geq 0$, or equivalently, $\varphi \in H_1^\infty(\mathcal{K}_{\Lambda^a, \mathcal{H}})$;
- (SC2) For every admissible kernel $k \in \mathcal{K}_{\Lambda^{na}}$, $([1] - \varphi \varphi^*) * k \geq 0$, or equivalently, $\varphi \in H_1^\infty(\mathcal{K}_{\Lambda^{na}, \mathcal{H}})$;

(AD) *There exists a positive kernels $\Gamma, \Gamma_1, \Gamma_2$ such that*

$$\begin{aligned} [1] - \varphi \varphi^* &= \Gamma * \prod_{j=1}^d ([1] - Z_j Z_j^*) \\ &= \Gamma_1 * \prod_{j \neq j_1} ([1] - Z_j Z_j^*) + \Gamma_2 * \prod_{j \neq j_2} ([1] - Z_j Z_j^*), \end{aligned}$$

where $Z_j(x) = \psi_j(x)$;

(TF) *There are unitary colligations Σ_a and Σ_{na} in the ample and nearly ample setting respectively, such that φ has transfer function representations*

$$\varphi = W_{\Sigma_a} = W_{\Sigma_{na}};$$

- (vN-a) $\varphi \in H^\infty(X, \mathcal{K}_{\Lambda^a, \mathcal{H}}) = H^\infty(X, \mathcal{K}_{\Lambda^{na}, \mathcal{H}})$ and for every representation π which is strictly contractive on the auxiliary test function σ_1 (respectively, the auxiliary test functions $\sigma_{\lambda_1}, \sigma_{\lambda_2}$), or contractive on these and either strongly or weakly continuous, we have $\|\pi(\varphi)\| \leq 1$;
- (vN-B) $\varphi \in H^\infty(X, \mathcal{K}_{\Lambda^a, \mathcal{H}}) = H^\infty(X, \mathcal{K}_{\Lambda^{na}, \mathcal{H}})$ and for every representation π which is a strict / strongly continuous / weakly continuous Brehmer representation with respect to Λ^a (respectively, Λ^{na}), we have $\|\pi(\varphi)\| \leq 1$.

In particular, the theorem implies that Λ^a and Λ^{na} are always equivalent preorderings.

Proof of Theorem 4.14. The idea is very much like that in the proof of Theorem 4.12. As we did there, we use the embedding ξ of X in the polydisk given in Lemma 2.17 and Theorem 4.6 to get a function $\tilde{\varphi}$ in $H_1^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$, which when restricted to the image of X under ξ pulls back to φ . Applying the polydisk realization theorem (Theorem 4.13) to $\tilde{\varphi}$, we obtain the equivalence of the various statements in the theorem over the polydisk, and then pulling back to X the result follows. \square

The Hilbert space \mathcal{H} is arbitrary in the last theorem, so we get the following corollary, generalizing Brehmer's theorem and a result in [30], via its obvious specialization to $H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$. Compare with Theorem 3.11.

Corollary 4.15. *Suppose $\Psi = \{\psi_1, \dots, \psi_d\}$ is a collection of test functions, Λ^{na} is a standard nearly ample preordering with maximal elements λ_1^m and λ_2^m . Let $\pi : A(\mathcal{K}_{\Lambda^{na}, \mathcal{L}(\mathcal{H})}) \rightarrow \mathcal{L}(\mathcal{H})$ be a Brehmer representation. Then π is completely contractive and so dilates to a completely contractive representation $\tilde{\pi}$ of the C^* -envelope of $A(\mathcal{K}_{\Lambda^{na}, \mathcal{L}(\mathcal{H})})$ with the property that it is the only completely positive agreeing with $\tilde{\pi}|_{A(\mathcal{K}_{\Lambda^{na}, \mathcal{L}(\mathcal{H})})}$.*

5. SOME APPLICATIONS

We give some more or less immediate applications of the material presented. For example, the following, which is the main result of [30], is the last corollary applied to the polydisk.

Corollary 5.1. *Let $\Psi = \{z_1, \dots, z_d\}$ be the coordinate functions on \mathbb{D}^d , Λ^{na} a standard nearly ample preordering with maximal elements λ_1^m and λ_2^m . Let $\pi : A(\mathbb{D}^d, \mathcal{L}(\mathcal{H})) \rightarrow \mathcal{L}(\mathcal{H})$ be a Brehmer representation. Then the contractions $\pi(z_j)$ dilate to a commuting unitary operators.*

Another interesting corollary of the realization theorem is a sort of weak form of the rational dilation property.

Corollary 5.2. *Let $\Psi = \{\psi_1, \dots, \psi_d\}$, $d \geq 2$, is a collection of test functions, Λ a standard ample preordering. Then any 2^{d-2} -contractive representation of $A(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ or weakly continuous 2^{d-2} -contractive representation of $H^\infty(X, \mathcal{K}_{\Lambda, \mathcal{H}})$ is completely contractive.*

Proof. This follows from Lemma 3.10, the last corollary and the fact that the auxiliary test functions with a standard nearly ample preordering are in $H^\infty(X, \mathcal{K}_{\Lambda, \mathbb{C}^{2^n}})$, $n \leq d - 2$. \square

On the polydisk, we then get the following.

Corollary 5.3. *For the polydisk \mathbb{D}^d , $d \geq 2$, any 2^{d-2} -contractive representation of $A(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$ or weakly continuous 2^{d-2} -contractive representation of $H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{H}))$ is completely contractive.*

This implies that 2-contractive representations of the tridisk algebra are completely contractive. In particular, examples like that due to Parrott of contractive representations of this algebra which are not completely contractive can only fail to be so by failing to be 2-contractive.

Corollary 5.4. *Any representation of $A(\mathbb{D}^3, \mathcal{L}(\mathcal{H}))$ or weakly continuous representation of $H^\infty(\mathbb{D}^3, \mathcal{L}(\mathcal{H}))$ which is 2-contractive is completely contractive. Equivalently, any such representation which is not completely contractive must fail to be 2-contractive.*

Here are a couple of other examples involving two test functions. Let X be an annulus \mathbb{A} with out boundary the unit circle \mathbb{T} and inner boundary $r\mathbb{T}$ for some $0 < r < 1$. Choose for test functions the set $\Psi = \{\psi_1(z) = z, \psi_2(z) = r/z\}$. By what we have shown (see also [37]), contractive representations of this algebra are automatically completely contractive, and so the rational dilation property holds. The rational dilation problem for the annulus was originally solved by Agler in [1] (see [25] for an alternate proof). It can be shown that although $A(X, \mathcal{K}_\Lambda)$ and $A(\mathbb{A})$ have different norms, they are the same algebra, and in fact as operator algebras they are completely boundedly equivalent [24] (see also [11]). One might naively expect that this would give yet another approach to solving this problem, but unfortunately it does not. Indeed, the same phenomenon occurs for multiply connected planar domains [24].

To perhaps better illustrate what might happen, consider instead the disk \mathbb{D} with test functions $\Psi = \{\psi_1(z) = z^2, \psi_2(z) = z^3\}$. This is an example of a constrained algebra, since $A(X, \mathcal{K}_\Lambda)$ consists of functions with first derivative equal to 0 at the origin. This algebra differs from the subalgebra of the disk algebra of functions with derivative 0 at the origin (that is, $\mathbb{C} + z^2A(\mathbb{D})$). For the latter, one can find examples of contractive representations which are not completely contractive (ie, rational dilation fails) [25], while for $A(X, \mathcal{K}_\Lambda)$, by what we have shown, it holds. Indeed, for $A(X, \mathcal{K}_\Lambda)$, a representation which maps the two test functions to contractions (satisfying the obvious constraint that $\pi(\psi_1)^3 = \pi(\psi_2)^2$) is completely contractive by Theorem 4.12 and Corollary 5.3, while there are examples of such representations of the constrained subalgebra of the disk algebra which are not even contractive (much as in the Kaijser-Varopoulos example) [25].

6. CONCLUSION

When we have more than two test functions over some set X , there will be preorderings with their associated algebras for which we cannot say much beyond what is in our initial realization theorem, Theorem 4.2. In particular, we do not know if the auxiliary test functions can be extended to matrix valued functions in our algebra, as we have in either the classical case or the cases of ample and nearly ample preorderings. We also wonder if there are other types of preorderings other than the ample and nearly ample ones which are equivalent.

It would be nice to know more concretely what the auxiliary test functions are, particularly over polydisks. The knowledge of this could provide a key tool in resolving a number of questions regarding the connection between Schur-Agler class in the classical sense and H^∞ over these domains, and hence resolving some of the mysteries surrounding these algebras. One immediate question is whether for $d > 3$ there are $(2^{d-2} - 1)$ -contractive representations which are not completely contractive (that is, are the bounds in Corollary 5.3 sharp?). Another is whether contractive representations of the polydisk algebras $A(\mathbb{D}^d)$ with $d \geq 3$ are necessarily completely bounded.

It would also be useful to know a norming set of boundary representations for the Agler algebra in the classical setting. Over the tridisk any representation sending the coordinate functions to a commuting tuples of unitaries are included (if we ignore irreducibility), but as we also saw in §3.4, other sorts of representations are also there. In the concrete examples given, these all send the coordinate functions to nilpotent partial isometries. As was mentioned, there will be examples of boundary representations which come from neither tuples of commuting unitaries, nor commuting tuples of nilpotent operators, but these in between cases are somewhat mysterious. Perhaps, up to unitary conjugation, they always send the coordinate functions to tuples of commuting partial isometries? However, from Kaijser-Varopoulos example, one sees that this alone is not enough. Note that the boundary representations coming from commuting unitaries are 1 dimensional, and our examples of nilpotent boundary representations are all finite dimensional. Is it the case that there is an upper bound to the dimension of all boundary representations? Though this seems unlikely, there is no obvious proof. There will be Schur-Agler class functions which peak on the boundary representations. Are they related to the polynomials from which these examples are initially drawn? In any case, for the nilpotent representations, these will presumably be polynomials of the same degree as the order of nilpotency.

What happens with the unit ball in \mathbb{C}^d , $d > 1$? It is well known that the unit ball of the Drury-Arveson algebra does not coincide with the unit ball of H^∞ of this space [6]. While one must be careful applying the results here in this setting since the test function is vector valued, it is still intriguing to speculate what algebras one might obtain with powers of the Drury-Arveson kernel.

Finally, the resemblance of results from real algebra to those presented here is striking. Are there some deeper connections? For example, could one use the techniques here to find, at least in some circumstances, a proof of such results as Schmüdgen's theorem?

REFERENCES

- [1] Jim Agler. Rational dilation on an annulus. *Ann. of Math. (2)*, 121(3):537–563, 1985.
- [2] Jim Agler. On the representation of certain holomorphic functions defined on a polydisc. In *Topics in operator theory: Ernst D. Hellinger memorial volume*, volume 48 of *Oper. Theory Adv. Appl.*, pages 47–66. Birkhäuser, Basel, 1990.
- [3] Jim Agler, John Harland, and Benjamin J. Raphael. Classical function theory, operator dilation theory, and machine computation on multiply-connected domains. *Mem. Amer. Math. Soc.*, 191(892):viii+159, 2008.
- [4] Jim Agler and John E. McCarthy. *Pick interpolation and Hilbert function spaces*, volume 44 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.
- [5] Jim Alger, John E. McCarthy, and N. J. Young. On the representation of holomorphic functions on polyhedra. *Michigan Math. J.*, 62(4):675–689, 2013.
- [6] Daniel Alpay and H. Turgay Kaptanoğlu. Some finite-dimensional backward-shift-invariant subspaces in the ball and a related interpolation problem. *Integral Equations Operator Theory*, 42(1):1–21, 2002.

- [7] C.-G. Ambrozie and D. Timotin. A von Neumann type inequality for certain domains in \mathbb{C}^n . *Proc. Amer. Math. Soc.*, 131(3):859–869 (electronic), 2003.
- [8] T. Andô. On a pair of commutative contractions. *Acta Sci. Math. (Szeged)*, 24:88–90, 1963.
- [9] J. Robert Archer. Positivity and the existence of unitary dilations of commuting contractions. In *The extended field of operator theory*, volume 171 of *Oper. Theory Adv. Appl.*, pages 17–35. Birkhäuser, Basel, 2007.
- [10] William Arveson. Subalgebras of C^* -algebras. III. Multivariable operator theory. *Acta Math.*, 181(2):159–228, 1998.
- [11] Catalin Badea, Bernhard Beckermann, and Michel Crouzeix. Intersections of several disks of the Riemann sphere as K -spectral sets. *Commun. Pure Appl. Anal.*, 8(1):37–54, 2009.
- [12] Bhaskar Bagchi, Tirthankar Bhattacharyya, and Gadadhar Misra. Some thoughts on Ando’s theorem and Parrott’s example. *Linear Algebra Appl.*, 341:357–367, 2002. Special issue dedicated to Professor T. Ando.
- [13] Sriram Balasubramanian. Non-commutative Carathéodory interpolation. *Integral Equations Operator Theory*, 68(4):529–550, 2010.
- [14] J. A. Ball, W. S. Li, D. Timotin, and T. T. Trent. A commutant lifting theorem on the polydisc. *Indiana Univ. Math. J.*, 48(2):653–675, 1999.
- [15] Joseph A. Ball, Animikh Biswas, Quanlei Fang, and Sanne ter Horst. Multivariable generalizations of the Schur class: positive kernel characterization and transfer function realization. In *Recent advances in operator theory and applications*, volume 187 of *Oper. Theory Adv. Appl.*, pages 17–79. Birkhäuser, Basel, 2009.
- [16] Joseph A. Ball and Vladimir Bolotnikov. Realization and interpolation for Schur-Agler-class functions on domains with matrix polynomial defining function in \mathbb{C}^n . *J. Funct. Anal.*, 213(1):45–87, 2004.
- [17] Joseph A. Ball and Moisés D. Guerra Huamán. Test functions, Schur-Agler classes and transfer-function realizations: the matrix-valued setting. *Complex Anal. Oper. Theory*, 7(3):529–575, 2013.
- [18] Joseph A. Ball and Dmitry S. Kaliuzhnyi-Verbovetskyi. Rational Cayley inner Herglotz-Agler functions: positive-kernel decompositions and transfer-function realizations. *Linear Algebra Appl.*, 456:138–156, 2014.
- [19] Joseph A. Ball and Tavan T. Trent. Unitary colligations, reproducing kernel Hilbert spaces, and Nevanlinna-Pick interpolation in several variables. *J. Funct. Anal.*, 157(1):1–61, 1998.
- [20] Stephen D. Barreto, B. V. Rajarama Bhat, Volkmar Liebscher, and Michael Skeide. Type I product systems of Hilbert modules. *J. Funct. Anal.*, 212(1):121–181, 2004.
- [21] David P. Blecher and Vern I. Paulsen. Explicit construction of universal operator algebras and applications to polynomial factorization. *Proc. Amer. Math. Soc.*, 112(3):839–850, 1991.
- [22] S. Brehmer. Über vertauschbare Kontraktionen des Hilbertschen Raumes. *Acta Sci. Math. Szeged*, 22:106–111, 1961.
- [23] M. J. Crabb and A. M. Davie. von Neumann’s inequality for Hilbert space operators. *Bull. London Math. Soc.*, 7:49–50, 1975.
- [24] R. G. Douglas and V. I. Paulsen. Completely bounded maps and hypo-Dirichlet algebras. *Acta Sci. Math. (Szeged)*, 50(1-2):143–157, 1986.
- [25] Michael A. Dritschel, Michael T. Jury, and Scott McCullough. Dilations and constrained algebras. *Oper. Matrices*, 2016. to appear, arXiv:1305.4272.
- [26] Michael A. Dritschel, Stefania Marcantognini, and Scott McCullough. Interpolation in semi-groupoid algebras. *J. Reine Angew. Math.*, 606:1–40, 2007.

- [27] Michael A. Dritschel and Scott McCullough. The failure of rational dilation on a triply connected domain. *J. Amer. Math. Soc.*, 18(4):873–918, 2005.
- [28] Michael A. Dritschel and Scott McCullough. Test functions, kernels, realizations and interpolation. In *Operator theory, structured matrices, and dilations*, volume 7 of *Theta Ser. Adv. Math.*, pages 153–179. Theta, Bucharest, 2007.
- [29] Michael A. Dritschel and Scott A. McCullough. Boundary representations for families of representations of operator algebras and spaces. *J. Operator Theory*, 53(1):159–167, 2005.
- [30] Anatolii Grinshpan, Dmitry S. Kaliuzhnyi-Verbovetskyi, Victor Vinnikov, and Hugo J. Woerdeman. Classes of tuples of commuting contractions satisfying the multivariable von Neumann inequality. *J. Funct. Anal.*, 256(9):3035–3054, 2009.
- [31] Anatolii Grinshpan, Dmitry S. Kaliuzhnyi-Verbovetskyi, and Hugo J. Woerdeman. Norm-constrained determinantal representations of multivariable polynomials. *Complex Anal. Oper. Theory*, 7(3):635–654, 2013.
- [32] Michael T. Jury. Universal commutative operator algebras and transfer function realizations of polynomials. *Integral Equations Operator Theory*, 73(3):305–329, 2012.
- [33] Michael T. Jury, Greg Knese, and Scott McCullough. Agler interpolation families of kernels. *Oper. Matrices*, 3(4):571–587, 2009.
- [34] Michael T. Jury, Greg Knese, and Scott McCullough. Nevanlinna-Pick interpolation on distinguished varieties in the bidisk. *J. Funct. Anal.*, 262(9):3812–3838, 2012.
- [35] Greg Knese. Kernel decompositions for Schur functions on the polydisk. *Complex Anal. Oper. Theory*, 5(4):1093–1111, 2011.
- [36] B. A. Lotto and T. Steger. von Neumann’s inequality for commuting, diagonalizable contractions. II. *Proc. Amer. Math. Soc.*, 120(3):897–901, 1994.
- [37] Meghna Mittal and Vern I. Paulsen. Operator algebras of functions. *J. Funct. Anal.*, 258(9):3195–3225, 2010.
- [38] W. Mlak. Positive definite contraction valued functions. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 15:509–512, 1967.
- [39] Paul S. Muhly and Baruch Solel. An algebraic characterization of boundary representations. In *Nonselfadjoint operator algebras, operator theory, and related topics*, volume 104 of *Oper. Theory Adv. Appl.*, pages 189–196. Birkhäuser, Basel, 1998.
- [40] Stephen Parrott. Unitary dilations for commuting contractions. *Pacific J. Math.*, 34:481–490, 1970.
- [41] James Pickering. Counterexamples to rational dilation on symmetric multiply connected domains. *Complex Anal. Oper. Theory*, 4(1):55–95, 2010.
- [42] Alexander Prestel and Charles N. Delzell. *Positive polynomials*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001. From Hilbert’s 17th problem to real algebra.
- [43] Mihai Putinar. Positive polynomials on compact semi-algebraic sets. *Indiana Univ. Math. J.*, 42(3):969–984, 1993.
- [44] Konrad Schmüdgen. The K -moment problem for compact semi-algebraic sets. *Math. Ann.*, 289(2):203–206, 1991.
- [45] Andrew T. Tomerlin. Products of Nevanlinna-Pick kernels and operator colligations. *Integral Equations Operator Theory*, 38(3):350–356, 2000.
- [46] N. Th. Varopoulos. On an inequality of von Neumann and an application of the metric theory of tensor products to operators theory. *J. Functional Analysis*, 16:83–100, 1974.

SCHOOL OF MATHEMATICS AND STATISTICS, HERSCHEL BUILDING, NEWCASTLE UNIVERSITY, NEWCASTLE UPON TYNE, NE1
7RU, UK
E-mail address: michael.dritschel@newcastle.ac.uk